Unit squared root test.

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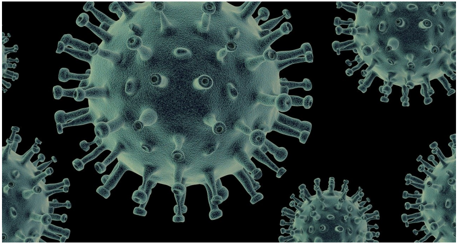

Consider the time series $(X_t)_{t\in\mathbb{Z}}$.

- We observe only n values $X_1, X_2, \ldots, X_n$ of this process, and we believe the observed values are only a part the process above lasting forever;
- We compute ACF $(\hat{\rho}_h)_{h\in\mathbb{Z}}$ and PACF $(\hat{\phi}^{YW}_{h,h})_{h\in\mathbb{Z}}$, and imagine that we have noticed that so many values $|\hat{\phi}^{YW}_{h,h}|$ are significant;
- In such a case, we no more believe this is a stationary time series, but we believe that the process of increments $X_t - X_{t-1}$ is stationary;
- In this purpose we construct the *unit square* test.

Two sided sequences

Let $\mathbb{Z}$ be the set of all integers, i.e. the natural numbers $\mathbb{N} = \{1, 2, ...\}$, their additive inverses $-\mathbb{N} = \{-1, -2, \dots\}$ and 0.

Two sided sequence

A *two sided* sequence is a sequence with infinite duration in the positive and negative directions. Formally two sided sequence is the transformation of $\mathbb{Z}$ into $\mathbb{R}$ with a generic element

$$x = (x_t)_{t \in \mathbb{Z}} = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots).$$

The set of all two sided sequences is denoted as $\mathbb{R}^{\mathbb{Z}}$.

Two sided sequences

The set of all two sided sequences form a vector space with the natural operations:

- for any vectors $(x_t)_{t\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(y_t)_{t\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$

$$(x_t)_{t\in\mathbb{Z}} + (y_t)_{t\in\mathbb{Z}} = (x_t + y_t)_{t\in\mathbb{Z}}$$

- for any $\alpha \in \mathbb{R}$ and any vector $(x_t)_{t\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$

$$\alpha(x_t)_{t\in\mathbb{Z}} = (\alpha x_t)_{t\in\mathbb{Z}}.$$

- the zero-vector is the sequence where all coordinates are 0:

$$\mathbf{0} = (0)_{t\in\mathbb{Z}}.$$

Useful linear operators

Shift-backward operators

For any $x \in \mathbb{R}^{\mathbb{Z}}$ we define the *shift-backward* operator $B : \mathbb{R}^{\mathbb{Z}} \mapsto \mathbb{R}^{\mathbb{Z}}$ as follows:

$$B(x_t)_{t\in\mathbb{Z}} = (x_{t-1})_{t\in\mathbb{Z}}.$$

We can alternatively express

$$Bx_t = x_{t-1} \quad \text{for any } t \in \mathbb{Z}.$$

Useful linear operators

Differentiation operator

For any $x \in \mathbb{R}^{\mathbb{Z}}$ we define the *differentiation* operator as follows:

$$\nabla (x_t)_{t\in\mathbb{Z}} = (x_t - x_{t-1})_{t\in\mathbb{Z}}.$$

We can alternatively express

$$\nabla x_t = x_t - x_{t-1} \quad \text{for any } t \in \mathbb{N}.$$

Useful linear operators

Properties:

- Both B and ∇ are linear operators on $\mathbb{R}^{\mathbb{Z}}$;
- The kernel of B is $\mathbf{0}$, hence B is an injective operator;
- The kernel of ∇ is the linear subspace of constant sequences $(x)_{t\in\mathbb{Z}}$, hence ∇ is not injective;
- If $\mathbf{I}$ is the *identity* operator on $\mathbb{R}^{\mathbb{Z}}$, then the following equation holds

$$\nabla x_t = x_t - x_{t-1} = \mathbf{I}x_t - Bx_t = (\mathbf{I} - B)x_t;$$

Polynomial operator

Let $B^2 := B \circ B$, $B^3 = B \circ B \circ B$ and more generally

$$B^n = \underbrace{B \circ B \circ \ldots \circ B}_{n \text{ times.}}.$$

Polynomial operator

The operator in on $\mathbb{R}^{\mathbb{Z}}$ with the form

$w(B) := \alpha_0 \mathbf{I} + \alpha_1 B + \ldots + \alpha_n B^n$ for some $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{N}, n \in \mathbb{N}$

is called a *polynomial operator.*

Polynomial operator

Characteristic polynomial

Let $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$. For any polynomial operator

$$w(B) = \alpha_0 \mathbf{I} + \alpha_1 B + \ldots + \alpha_n B^n$$

we define the corresponding real (or even complex) polynomial

$$w(\lambda) = \alpha_0 + \alpha_1 \lambda + \ldots + \alpha_n \lambda^n$$

called *characteristic polynomial*.

Polynomial operator and its characteristic polynomial

Theorem

The polynomial operator $w(B)$ is *invertible* on $\mathbb{R}^{\mathbb{Z}}$ if and only if its characteristic polynomial $w(\lambda)$ has all roots outside the unit sphere in the complex plane. That is the following equation holds:

$$\forall_{\lambda \in \mathbb{C}} w(\lambda) = 0 \Rightarrow |\lambda| > 1.$$

Examples

Example

The following operator

$$w(B) = \mathbf{I} - \phi B$$

is invertible on $\mathbb{R}^{\mathbb{Z}}$ if and only if $|\phi| < 1$. Its characteristic polynomial is

$$w(\lambda) = 1 - \phi\lambda.$$

A unique root is $\lambda_1 = \frac{1}{\phi}$ for which $|\lambda| = \frac{1}{|\phi|}$. Then

- if $|\phi| > 1$ then $|\lambda| < 1$ and by the last theorem, $w(B)$ is not invertible;
- if $|\phi| < 1$ then $|\lambda| > 1$ and by the last theorem, $w(B)$ is invertible;

Examples

Example

The following operator:

$$w(B) = 2\mathbf{I} - 3B + B^2$$

is not invertible. If it is invertible then the moduli of all roots of the characteristic polynomial

$$w(\lambda) = 2 - 3\lambda + \lambda^2$$

are both strictly greater than 1. But it is not the case. Both square roots are $\lambda_1 = 1$ and $\lambda_2 = 2$ and one of it satisfies $|\lambda_1| = 1$ (not > 1).

Processes AR, MA

The following processes can be expressed with the polynomial from

- Processes AR(p) can be expressed in the following form

  $$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = \epsilon_t,$$

  hence it can be expressed as

  $$w(B)X_t = \epsilon_t$$

  with the polynomial operator

  $$w(B) = (\mathbf{I} - \phi_1 B - \ldots - \phi_p B^p);$$

- Processes MA(q) can be expressed in the following form

  $$X_t = \epsilon_t - \theta_1 \epsilon_{t-1} - \ldots - \theta_q \epsilon_{t-q},$$

  hence it can be expressed as

  $$X_t = \pi(B)\epsilon_t$$

  with the polynomial operator

  $$\pi(B) = (\mathbf{I} - \theta_1 B - \ldots - \theta_q B^p).$$

Process ARMA

- Finally ARMA(p,q) can be expressed in the following form

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = \epsilon_t - \theta_1 \epsilon_{t-1} - \ldots - \theta_q \epsilon_{t-q},$$

hence it can be expressed as follows

$$w(B) X_t = \pi(B) \epsilon_t,$$

where

$$w(B) = (\mathbf{I} - \phi_1 B - \ldots - \phi_p B^p);$$

and

$$\pi(B) = (\mathbf{I} - \theta_1 B - \ldots - \theta_q B^q).$$

Stationarity of ARMA

Theorem on stationarity of ARMA

The processes ARMA(p,q) in the polynomial form

$$w(B)X_t = \pi(B)\epsilon_t$$

is stationary if and only if its characteristic polynomial w satisfies

$$\forall_{\lambda \in \mathbb{C}} w(\lambda) = 0 \Rightarrow |\lambda| > 1.$$

Main corollaries

Corollaries

The process ARMA(p,q) in the form $w(B)X_t = \pi(B)\epsilon_t$

- is stationary if and only the characteristic polynomial w has all roots outside the unit circle in $\mathbb{C}$;
- the polynomial operator is invertible, hence the inverse operator $w^{-1}(B)$ exists;
- there is a power series of $w^{-1}(B)$ with the same coefficients as the rational function $\frac{1}{w(\lambda)}$;
- in such a case the time series may be expressed explicitly as follows:

$$X_t = w^{-1}(B)\pi(B)\epsilon_t;$$

Nonstationarity of AR(1)

Consider the model AR(1) with included constant μ:

$$X_t - \mu = \phi(X_{t-1} - \mu) + \epsilon_t,$$

and assume additionally $\phi \in (-1, 1]$. Equivalently we may express

$$X_t - X_{t-1} = \mu(1 - \phi) + (\phi - 1)X_{t-1} + \epsilon_t,$$

or

$$\nabla X_t = \underbrace{\mu(1 - \phi)}_{\phi_0^*} + \underbrace{(\phi - 1)}_{\phi_1^*} BX_t + \epsilon_t.$$

By the current results, X_t is nonstationary if and only $\phi = 1$ or equivalently $\phi_0^* = \phi_1^* = 1$.

Unit root test for AR(1)

We test the following equivalent hypothesis

- X_t is nonstationary;
- $\nabla X_t = X_t - X_{t-1}$ yields a white noise, that is X_t is the standard *random walk*;
- $\phi = 1$;
- $\phi_0^* = \phi_1^* = 0$.

Unit root test for AR(1)

More formally,

- We test the hypothesis that X_t is not stationary, against X_t it is nonstationary.
- For this purpose we test the following auxiliary hypothesis:

$$H_0 : \phi = 1 \quad \text{VS} \quad H_1 : \phi < 1.$$

- Equivalently we test

$$H_0 : \phi_0^* = \phi_1^* = 0 \quad \text{VS} \quad H_1 : \phi_1^* < 0;$$

- For this purpose we apply the *Dickey-Fuller* test called *unit root* test.

Unit root test for AR(1) - construction

- We construct the estimator of $[\phi_0^*, \phi_1^*]^T$ using the *least squared method*, i.e. the estimator $[\hat{\phi}_0^*, \hat{\phi}_1^*]^T$ minimizes the following expression

$$\sum_{t=1}^{n} (\nabla X_t - \phi_0^* - \phi_1^* B X_t)^2 \quad \text{for } \phi_0^*, \phi_1^* \in \mathbb{R}.$$

- Then $\hat{\phi}_1^*$ is a random variable whose variance is approximated by the following estimator:

$$Var(\hat{\phi}_1^*) \approx \frac{\frac{1}{n-3} \sum_{t=2}^{n} (\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* B X_t)^2}{\sum_{t=2}^{n} (B X_t - \overline{X})^2} \quad \text{where} \quad \bar{X} = \frac{1}{n} \sum_{t=1}^{n} X_t.$$

Unit root test for AR(1) - construction

- Let us denote:

$$\hat{Var}(\hat{\phi}_1^*) = \frac{\frac{1}{n-3}\sum_{t=2}^{n}(\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* BX_t)^2}{\sum_{t=2}^{n}(BX_t - \overline{X})^2} \quad \text{and} \quad \hat{SE}(\hat{\phi}_1^*) = \sqrt{\hat{Var}(\hat{\phi}_1^*)}.$$

- The relative standard error of $\hat{\phi}_1^*$ is the test statistic of the **unit root** test

$$DF = \frac{\hat{\phi}_1^*}{\hat{SE}(\hat{\phi}_1^*)}.$$

- If the hypothesis H_0 is true, then the asymptotic statistic DF has the distribution tabularized by Dickey and Fuller.

Unit root test for AR(p) - construction

Let us generalize the problem of nonstationarity toward $AR(p)$:

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \ldots + \phi_p(X_p - \mu) + \epsilon_t.$$

- Its characteristic polynomial has the following form

$$w(\lambda) = 1 - \phi_1\lambda - \phi_2\lambda^2 - \ldots - \phi_p\lambda^p.$$

- If $\lambda = 1$ is one of the roots of w, by our theorems X_t is not stationary, and it is a kind of random walk.
- Because of that, we test the auxiliary hypothesis

$$H_0 : w(1) = 0 \quad \text{VS} \quad H_0 : w(1) \neq 0.$$

Unit root test for AR(p) - construction

- Since
$$w(1) = 0 \Leftrightarrow 1 - \phi_1 - \phi_2 - \ldots - \phi_p = 0,$$
- we test equivalently
$$H_0 : \sum_{k=1}^{p} \phi_k = 1 \quad \text{VS} \quad H_0 : \sum_{k=1}^{p} \phi_k \neq 1.$$
- Let us transform this hypothesis.

Unit root test for AR(p) - construction

The expression

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \ldots + \phi_p(X_p - \mu) + \epsilon_t.$$

is equivalent to

$$\nabla X_t = \phi_0^* + \phi_1^* B X_{t-1} + \phi_2^* \nabla X_{t-1} + \ldots + \phi_p^* \nabla X_{t-(p-1)} + \epsilon_t,$$

where

$$\phi_0^* = \mu \left(1 - \sum_{i=1}^{p} \phi_i \right), \quad \phi_1^* = \sum_{i=1}^{p} \phi_i - 1$$

and for $j = 2, 3, \ldots, p$

$$\phi_j^* = -\sum_{i=j}^{p} \phi_i.$$

Unit root test for AR(p) - construction

- Hence the hypothesis

$$H_0 : \sum_{k=1}^{p} \phi_k = 1 \quad \text{VS} \quad H_0 : \sum_{k=1}^{p} \phi_k \neq 1;$$

is equivalent to

$$H_0 : \phi_0^* = \phi_1^* = 0 \quad \text{VS} \quad H_0 : \phi_0^* \neq 0 \text{ or (or even and) } \phi_1^*;$$

- As a result, if H_0 is true then the differentiated process ∇X_t is the AR(p-1). Indeed

$$\nabla X_t = \underbrace{\phi_0^* + \phi_1^* B X_{t-1}}_{=0 \text{ under } H_0} + \phi_2^* \nabla X_{t-1} + \ldots + \phi_p^* \nabla X_{t-(p-1)} + \epsilon_t .$$

Unit root test for AR(p) - construction

We test the following hypothesis at the same time

- X_t is non-stationary;
- $\nabla X_t = X_t - X_{t-1}$ is AR(p-1) (without included constant);
    - In case of $AR(1)$, ∇X_t is a white noise;
    - Generally, ∇X_t reduces the lag from p by 1;
- $w(1) = \sum_{k=1}^{p} \phi_k = 1$;
- $\phi_0^* = \phi_1^* = 0$.

Unit root test for AR(p) - construction

We construct the test statistic.

- The estimator of $[\phi_0^*, \phi_1^*, \ldots, \phi_p^*]^T$ is computed by the least squared equations i.e. find $[\hat{\phi}_0^*, \hat{\phi}_1^*, \ldots, \hat{\phi}_p^*]^T$ that minimizes

$$\sum_{t=1}^{p} (\nabla X_t - \phi_0^* - \phi_1^* B X_{t-1} - \phi_2^* \nabla X_{t-1} - \ldots - \phi_p^* \nabla X_{t-(p-1)})^2 \quad \text{for } \phi_0^*, \phi_1^*, \ldots, \phi_p^* \in \mathbb{R}.$$

- The relative standard error of $\hat{\phi}_1^*$ is the test statistic of the **unit root** test

$$DF = \frac{\hat{\phi}_1^*}{\hat{SE}(\hat{\phi}_1^*)}.$$

- If the hypothesis H_0 is true, then the asymptotic statistic DF has the distribution tabularized by Dickey and Fuller (the same as in case of AR(1)).

Summary

- We use the *unit root* test when we are skeptical about the issue of stationarity;
- We believe however that the differentiation yields the stationary time series in AR(p-1);
- An important premise for applying this test is many significant values of PACF.