Yule Walker formula.

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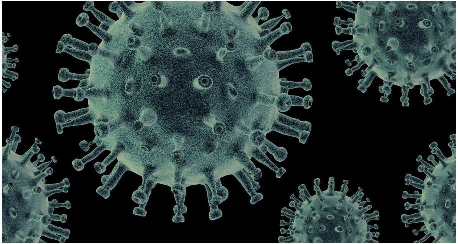

Definicja 1 (Time series)

Time series *X_t is a family of random variables enumerated by $\mathbb{Z}$ (or $\mathbb{N}$ or sometimes $\mathbb{N}_0$). This enumeration assigns uniquely determined order, in particular the index of any observation determines its moment of time.*

Definicja 2

White noise *ϵ_t is a time series satisfying:*

- $E\epsilon_t = 0$,
- $Cov(\epsilon_t, \epsilon_{t+h}) = 0$ *dla* $h \neq 0$,
- $Cov(\epsilon_t, \epsilon_t) = Var(\epsilon_t) = \sigma^2$ $(t \in \mathbb{Z})$;

Time series on the base of white noise

Assume

- X_t jest adapted to the filtration $\mathscr{F}_t = \sigma(\epsilon_t, \epsilon_{t-1}, \ldots)$;
- $\mathscr{F}_t$ is interpretated as an information about the history up to time t;
- More precisely, the history up to t is stored in $\{\epsilon_\tau : \tau \leq t\}$ is a key to mystery why we observe X_t:
- Mathematically we can write $X_t = \varphi_t(\epsilon_t, \epsilon_{t-1}, \ldots)$;

Stationary time series

The time series is called **stationary** if belongs to $\mathscr{L}^2(\mathbb{P})$ and

- $\mu_t = EX_t$ is time invariant (we write then μ);
- $\gamma_{t,h}$ depends only on h (we write then γ_h).

Stationary time series type AR(p)

Today we consider the time series AR(p) in the form:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \epsilon_t,$$

where $\phi_1, \phi_2, \ldots, \phi_p$ are unknown parameters. Simple example shows that AR(p) may be stationary or not:

- the model $AR(1)$ in the form $X_t = X_{t-1} + \epsilon_t$ is not stationary;
- the model $AR(1)$ in the form $X_t = \phi X_{t-1} + \epsilon_t$ is stationary if and only if $\phi \in (-1, 1)$;

- Assume that our model AR(p) is stationary.
- We find the preliminary estimator of $\phi_1, \phi_2, \ldots, \phi_p$ using Yule-Walker formula.

The Yule-Walker formula

Theorem (Yule-Walker formula)

Let X_t be a stationary time series AR(p) in the following form

$$X_t = \sum_{k=1}^{p} \phi_p X_{t-k} + \epsilon_t$$

in which γ_h is its autocovariance function and ρ_h its autocorrelation for $h \in \mathbb{N} \cup \{0\}$. Then the following equality holds:

$$\gamma_h = \sum_{k=1}^{p} \phi_k \gamma_{h-k}, \text{ and } \rho_h = \sum_{k=1}^{p} \phi_k \rho_{h-k} \forall_{h \in \mathbb{N}_0}.$$

Yule-Walker formula - derivation

The derivation of Yule-Walker formula

Let us write AR(p) as:

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + \epsilon_t$$

and assume its stationarity. For $h > 0$ we have

$$Cov(X_{t-h}, X_t) = Cov\left(X_{t-h}, \sum_{k=1}^{p} \phi_k X_{t-k} + \epsilon_t\right),$$

By the bilinearity of the covariance and by $Cov(X_{t-h}, \epsilon_t) = 0$ we have

$$\underbrace{Cov(X_{t-h}, X_t)}_{=\gamma_{-h}} = \sum_{k=1}^{p} \phi_k \underbrace{Cov\left(X_{t-h}, X_{t-k}\right)}_{=\gamma_{-(h-k)}}(VERTE).$$

The derivation of Yule-Walker formula-continue

By the stationarity X_t we have (for any h)

$$\gamma_h = Cov(X_{t+h}, X_t) = Cov(X_{t+h}, X_{t+h-h}) = Cov(X_t, X_{t-h}) = \gamma_{-h}$$

and hence

$$\underbrace{\gamma_h}_{=\gamma_{-h}} = \sum_{k=1}^{p} \phi_k \underbrace{\gamma_{h-k}}_{=\gamma_{-(h-k)}} .$$

After dividing over γ_0 we have

$$\rho_h = \sum_{k=1}^{p} \phi_k \rho_{h-k}.$$

Hennce we obtain the Yule-Walker formula.

The consequence of Yule-Walker formulas -the preliminary estimation of parameters

Let us write the Yule-Walker formula for $h = 1, 2, \ldots, p$ remembering that $\rho_0 = 1$:

$$\begin{cases} \rho_1 = & \phi_1 & +\phi_2\rho_{-1} & \cdots & +\phi_{p-1}\rho_{-(p-2)} & +\phi_p\rho_{-(p-1)} \\ \rho_2 = & \phi_1\rho_1 & +\phi_2 & \cdots & +\phi_{p-1}\rho_{-(p-2)} & +\phi_p\rho_{-(p-2)} \\ \vdots & & & & & \\ \rho_p = & \phi_1\rho_{p-1} & +\phi_2\rho_{p-2} & \cdots & +\phi_{p-1}\rho_1 & +\phi_p \end{cases}$$

The consequence of Yule-Walker formulas -the preliminary estimation of parameters

From $\rho_{-h} = \rho_h$ we have the system of linear equations:

$$\underbrace{\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_p \end{bmatrix}}_{\tilde{\rho}} = \underbrace{\begin{bmatrix} 1 & \rho_1 & \rho_2 & \dots & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \dots & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \dots & \rho_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \dots & 1 \end{bmatrix}}_{\mathbf{R}_\rho} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_p \end{bmatrix}}_{\phi}$$

Remarks:

1. The matrix $\mathbf{R}_\rho$ - of correlations of the vector $(X_1, X_2, \dots, X_n)$ is positively definite, hence nonsingular;
2. Hence there exists an inverse matrix $\mathbf{R}_\rho^{-1}$;
3. The vector ϕ is uniquely determined $\phi = \mathbf{R}_\rho^{-1}\tilde{\rho}$, i.e. only such a vector determines a stationary model type AR(p);

The preliminary estimation of parameters of stationary time series AR(p)

Since we observe $X_1, X_2, \ldots, X_n$ we can obtain the sampling autocorrelations:

$$\hat{\rho}_h = \frac{\hat{\gamma}_h}{\hat{\gamma}_0},$$

where for $h \geq 0$ we have

$$\hat{\gamma}_h = \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \bar{X})(X_{t+h} - \bar{X})$$

and

$$\bar{X} = \frac{1}{n} \sum_{t=1}^{n} X_t.$$

The preliminary estimation of parameters of stationary time series AR(p)

Substituting ρ_h by its estimator $\hat{\rho}_h$ the Yule-Walker furmula provide

$$\underbrace{\begin{bmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \hat{\rho}_3 \\ \vdots \\ \hat{\rho}_p \end{bmatrix}}_{\hat{\tilde{\rho}}} = \underbrace{\begin{bmatrix} 1 & \hat{\rho}_1 & \hat{\rho}_2 & \dots & \hat{\rho}_{p-1} \\ \hat{\rho}_1 & 1 & \hat{\rho}_1 & \dots & \hat{\rho}_{p-2} \\ \hat{\rho}_2 & \hat{\rho}_1 & 1 & \dots & \hat{\rho}_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}_{p-1} & \hat{\rho}_{p-2} & \hat{\rho}_{p-3} & \dots & 1 \end{bmatrix}}_{\mathbf{R}_{\hat{\rho}}} \underbrace{\begin{bmatrix} \hat{\phi}_1^{YW} \\ \hat{\phi}_2^{YW} \\ \hat{\phi}_3^{YW} \\ \vdots \\ \hat{\phi}_p^{YW} \end{bmatrix}}_{\hat{\phi}^{YW}}$$

Multiplying both sides by the corresponding inverse matrix we have the **preliminary estimator** of $\phi_1, \phi_2, \dots, \phi_p$,

$$\hat{\phi}^{YW} = \mathbf{R}_{\hat{\rho}}^{-1} * \hat{\tilde{\rho}}.$$

The preliminary estimation of parameters of stationary time series AR(p)

The $\hat{\phi}^{YW}$ is a preliminary estimator of ϕ and cannot be a final estimation due to the following reasons:

1. The model AR(p) is not always stationary, in practice the assumption of stationarity is not reasonal;
2. Even if the model AR(p) is stationary $\hat{\phi}^{P}$ is sensitive to rounding errors, especially if the parameters of AR(p) are close to the *edge* of stationarity;
3. But because we ususlly are not aware the value of p, $\hat{\phi}^{YW}$ is usefull as a preliminary estimator, which we apply for determining p (the details later);
4. The final estimator of ϕ will be determined using the *maximum likelihood method* method.

Yule Walker formula- special cases

Consider the model $AR(1)$ in the form:

$$X_t = \phi X_{t-1} + \epsilon_t$$

for $\phi \in (-1, 1)$. Then Yule Walker formula is a simple equation

$$\rho_1 = \phi \rho_0$$

hence $\phi = \rho_1$. Substituting ρ_1 by its estimator $\hat{\rho}_1$ we have

$$\hat{\phi}^{YW} = \hat{\rho}_1.$$

Yule Walker formula- special cases

Consider the model $AR(2)$ in the form:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

Then Yule Walker formula has the following form:

$$\rho_h = \phi_1 \rho_{h-1} + \phi_2 \rho_{h-2}$$

for $h = 1, 2$ or in matrix form

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Yule Walker formula- special cases

Computing corresponding determinants we have

$$\begin{cases} \phi_1 = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2} \\ \phi_2 = \frac{\rho_2-\rho_1^2}{1-\rho_1^2} \\ \end{cases}$$

Substituting ρ_1 and ρ_2 by $\hat{\rho}_1$ and $\hat{\rho}_2$ we obtain then

$$\begin{cases} \hat{\phi}_1^{YW} = \frac{\hat{\rho}_1(1-\hat{\rho}_2)}{1-\hat{\rho}_1^2} \\ \hat{\phi}_2^{YW} = \frac{\hat{\rho}_2-\hat{\rho}_1^2}{1-\hat{\rho}_1^2} \end{cases}$$