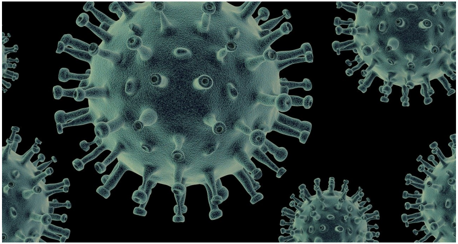
Stationary time series.

May 6, 2021



Definicja 1 (Time series)

Time series *X_t is a family of random variables enumerated by $\mathbb{Z}$ (or $\mathbb{N}$ or sometimes $\mathbb{N}_0$).*

Definicja 2

White noise *ϵ_t is a time series satisfying:*

- $E\epsilon_t = 0$,
- $Cov(\epsilon_t, \epsilon_{t+h}) = 0$ *dla* $h \neq 0$,
- $Cov(\epsilon_t, \epsilon_t) = Var(\epsilon_t) = \sigma^2$ $(t \in \mathbb{Z})$;

Time series on the base of white noise

Assume

- X_t jest adapted to the filtration $\mathscr{F}_t = \sigma(\epsilon_t, \epsilon_{t-1}, \ldots)$;
- $\mathscr{F}_t$ is interpretated as an information about the history up to time t;
- More precisely, the history up to t is stored in $\{\epsilon_\tau : \tau \leq t\}$ is a key to mystery why we observe X_t:
- Mathematically we can write $X_t = \varphi_t(\epsilon_t, \epsilon_{t-1}, \ldots)$;

The problem

The statistician

- knows the realization $X_1, X_2, \ldots, X_n$;
- do not know the realization ϵ_t and does not know σ^2 ;
- do not know φ_t but guess its existence;
- perhaps knows the class of the functions, e.g. a polynomial with unknown coefficients plus white noise;
- try to estimate φ_t on the base of observations $X_1, X_2, \ldots, X_n$;

- **The expected value function:**

$$\mu_t = EX_t;$$

- **The autocovariance function:**

$$\gamma_{t,h} := Cov(X_t, X_{t+h}).$$

From autocovariance we can obtain:

- **The function of variance:**

$$\sigma_t^2 := \gamma_{t,0} = Cov(X_t, X_t) = Var(X_t)$$

- **The autocorrelation function:**

$$\rho_{t,h} := Corr(X_t, X_{t+h}) = \frac{\gamma_{t,h}}{\sqrt{\sigma_t^2 \sigma_{t+h}^2}}.$$

for $t, h \in \mathbb{Z}$.

Problems in estimation

- We cannot estimate μ_t, σ_t^2, $\gamma_{t,h}$, $\rho_{t,h}$ on the base of sampling, because we know only one realization of time series and this realization is incomplete;
- We need to estimate these parameters at distinct time variables t: we estimate $\mu_1, \mu_2, \ldots, \mu_n$;
- The variables X_t are not independent.

Stationary time series

Stationary time series

The time series is called **stationary** if belongs to $\mathscr{L}^2(\mathbb{P})$ and

- $\mu_t = EX_t$ is time invariant (we write then μ);
- $\gamma_{t,h}$ depends only on h (we write then γ_h).

For stationary time series the following parameters are time invariant as well:

- the variance function

$$\sigma_t^2 = \gamma_{t,0} = \gamma_0 := const,$$

  and the variance is called σ^2,
- The autocorrelation function

$$\rho_{t,h} = \frac{\gamma_{t,h}}{\sqrt{\sigma_t^2 \sigma_{t+h}^2}} = \frac{\gamma_h}{\sigma^2} = \frac{\gamma_h}{\gamma_0} = const,$$

  and the autocorrelation function we denote as ρ_h.

The autocovariance and autocorrelation functions are both even

Since for any random vector (X, Y) it holds:

$$Cov(X, Y) = Cov(Y, X)$$

hence for stationary time series, for $h \in \mathbb{Z}$ it holds

$$\gamma_h = Cov(X_t, X_{t+h}) = Cov(X_{t+h}, X_t) = Cov(X_{t+h}, X_{t+h-h}) = \gamma_{-h},$$

and similarly

$$\rho_h = \rho_{-h}.$$

From Schwartz inequality it follows that:

$$|Cov(X,Y)| \le \sqrt{Var(X)Var(Y)} \quad \text{and} \quad |Corr(X,Y)| \le 1$$

hence for stationary time series, for $h \in \mathbb{Z}$ it holds

$$|\gamma_h| = |Cov(X_t, X_{t+h})| \le \sigma^2 \quad \text{and} \quad |\rho_h| = |Corr(X_t, X_{t+h})| \le 1.$$

The autocovariance and autocorrelation functions

Basic properties of autocovariance and autocorrelation functions

For stationary time series $(X_t)_{t\in\mathbb{Z}}$ the autocovariance and the autocorrelation functions obey the following properties:

- for $h \in \mathbb{Z}$:
$$\gamma_h = \gamma_{-h} \quad \text{and} \quad \rho_h = \rho_{-h};$$
- for $h \in \mathbb{Z}$:
$$|\gamma_h| \leq \sigma^2 \quad \text{and} \quad \rho_h \leq 1;$$
- and for $h = 0$ the inequalities above are in fact equalities:
$$\gamma_0 = \sigma^2 \quad \text{and} \quad \rho_0 = 1.$$

For the expected value μ the natural estimator is the **arithmetic mean** $\bar{X}$.

The arithmetic mean

The arithmetic mean from the observed realization of time series $(X_1, X_2, \ldots, X_n)$ is

$$\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t.$$

For the variance σ the natural estimator is the **empirical variance** $\hat{\sigma}^2$.

The sampling variance

The sampling variance from observed realization of time series $(X_1, X_2, \ldots, X_n)$ is:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{t=1}^{n} \left(X_t - \bar{X}\right)^2 .$$

For the autocovariance function γ_h the natural estimator is the **empirical autocovariance function** $\hat{\gamma}_h$.

The empirical autocovariance

The empirical autocovariance from observed realization of time series $(X_1, X_2, \ldots, X_n)$ is:

$$\hat{\gamma}_h := \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \bar{X})(X_{t+h} - \bar{X}).$$

For the autocorrelation function ρ_h the natural estimator is the **empirical autocorrelation function**.

The empirical autocorrelation function

The empirical autocorrelation function from observed realization of time series $(X_1, X_2, \ldots, X_n)$ is:

$$\hat{\rho}_h := \frac{\hat{\gamma}_h}{\hat{\sigma}^2} = \frac{\hat{\gamma}_h}{\hat{\gamma}_0}.$$

The estimators of parameters of stationary time series - summary

Formula	Estimator
$\mu = EX_t$	$\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$
$\sigma = Var(X_t)$	$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n}(X_t - \bar{X})^2$
$\gamma_h = Cov(X_t, X_{t+h})$	$\hat{\gamma}_h = \frac{1}{n}\sum_{t=1}^{n-h}(X_t - \bar{X})(X_{t+h} - \bar{X})$
$\rho_h = Corr(X_t, X_{t+h})$	$\hat{\rho}_h = \frac{\hat{\gamma}_h}{\hat{\sigma}^2}$

Stationary time series

- We can preliminarily assume that the time series is stationary in order to identify preliminarily the model and its parameters;
- Example of time series:
  - the white noise ϵ_t;
  - All models of *moving average* of type $MA(q)$:

$$X_t = \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \ldots - \theta_q \epsilon_{t-q};$$

    $\theta_1, \theta_2, \ldots, \theta_q$ unknown parameters;
  - some autoregressive models of type $AR(p)$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \epsilon_t;$$

    $\phi_1, \phi_2, \ldots, \phi_p$ unknown parameters;
  - some mixed models of type $ARMA(p, q)$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \ldots - \theta_q \epsilon_{t-q};$$

    $\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$ unknown parameters;

- It is easy to verify that $MA(q)$ is stationary (see the next slides);
- But this property $AR(p)$ and $ARMA(p,q)$ requires depth analysis (more later).

The stationarity of white noise

The stationarity of white noise is easy to verify:

- Note that
$$E\epsilon_t = 0 \quad \text{for all } t \in \mathbb{Z},$$
hence $\mu_t = \mu = 0$, and μ_t **does not depend on** t,
- and
$$Cov(X_t, X_{t+h}) = \delta_h := \begin{cases} 0 & if \quad h \neq 0 \\ \sigma^2 & if \quad h = 0. \end{cases}$$
Hence $\gamma_{t,h} = \gamma_h = \delta_h$ **not depending on** t.

The stationarity of MA(q)

The model $MA(q)$ has the following form:

$$X_t = \epsilon_t - \sum_{k=1}^{q} \theta_k \epsilon_{t-k}.$$

Obviously, by definition of white noise $EX_t = 0$. It is easy to see that

$$\gamma_0 = Cov(X_t, X_t) = Var\left(\epsilon_t - \sum_{k=1}^{q} \theta_k \epsilon_{t-k}\right) = \sigma^2 \left(1 + \sum_{k=1}^{q} \theta_k^2\right),$$

hence is invariant in t. Let $h \geq 1$. By the bilinearity of covariance we have

$$Cov(X_t, X_{t+h}) = Cov\left(\epsilon_t - \sum_{k=1}^{q} \theta_k \epsilon_{t-k}, \epsilon_{t+h} - \sum_{k=1}^{q} \theta_k \epsilon_{t+h-k}\right) (VERTE)$$

The stationarity of MA(q) (continue)

$$= -\sum_{k=1}^{q} \theta_k Cov(\epsilon_t, \epsilon_{t+h-k})$$

$$+ \sum_{k=1}^{q} \sum_{k'=1}^{q'} \theta_k \theta_{k'} Cov(\epsilon_{t-k}, \epsilon_{t+h-k'})$$

Since for the white noise we have $Cov(\epsilon_i, \epsilon_j) = \delta_{i-j}$ $(\forall_{i,j})$ we have

$$= -\sum_{k=1}^{q} \theta_k \delta_{h-k} + \sum_{k=1}^{q} \sum_{k'=1}^{q} \theta_k \theta_{k'} \delta_{k'-k-h}.$$

We have proven that $\gamma_{t,h}$ does not depend on t for $h > 0$. But if we further transform this formula we have $\gamma_{t,t+h} = 0$ for $h > q$. For $h = 1, 2, \ldots, q-1$ we have

$$\gamma_{t,h} = -\theta_h + \sum_{k=1}^{q-h} \theta_k \theta_{k+h},$$

(VERTE)

The stationarity of MA(q) (continue)

and $\gamma_{t,q} = -\theta_q$. For $h < 0$ the formula is the same. Finally

$$\gamma_{t,h} = \gamma_h = \begin{cases} 0 & \text{if} \quad h \leq -q-1 \text{ lub } h \geq q+1 \\ -\theta_q & \text{if} \quad h = -q \text{ lub } h = q \\ -\theta_h + \sum_{k=1}^{q-h} \theta_k \theta_{k+h} & \text{if} \quad -(q-1) \leq h \leq q-1 \end{cases}$$

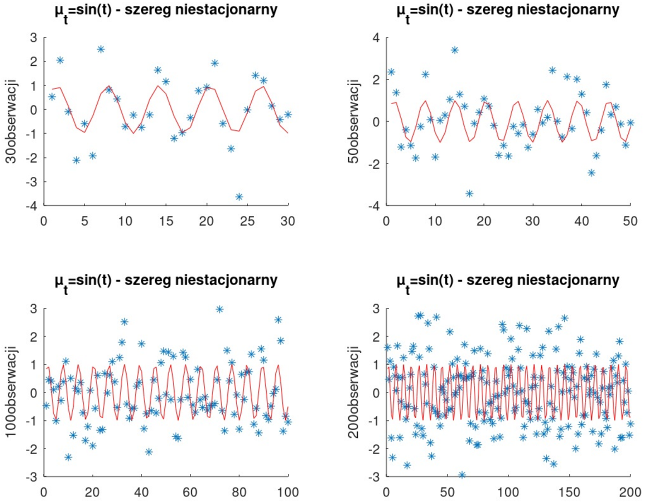
Hence the series of type MA(q) is stationary.

The trajectory of stationary time series

- The visual diagnosis of stationarity is difficult, the trajectory poorly helps us to recognize this property;
- But the trajectory should oscillate around a fixed value (since $\mu_t = const$);
- The variety distance should be time invariant ($\sigma_t^2 = const$);
- But around a constant may oscillate another deterministic curve non-constant: e.g. the sinusoid line;
- Similar situation with the varianceą;
- See next slides.

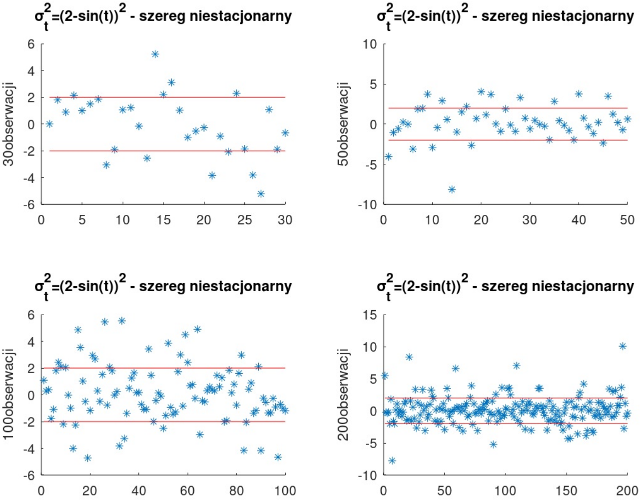
The example of non-stationary time series μ_t depending on t

The trajectory $X_t = \sin(t) + \epsilon_t$ seems to oscillate around the horizontal axes 0, but in fact oscillates around the sinusoid.



The example of non-stationary time series σ_t^2 depending on t

The trajectory $X_t = (2 + \sin(t))\epsilon_t$ does not detect visually the time variety of variance.



The transition between the stacjonarity and non-stationarity is fine.

The transition between the stacjonarity and non-stationarity is thin: For $t \in \mathbb{N}$ the time series is called the *random walking*

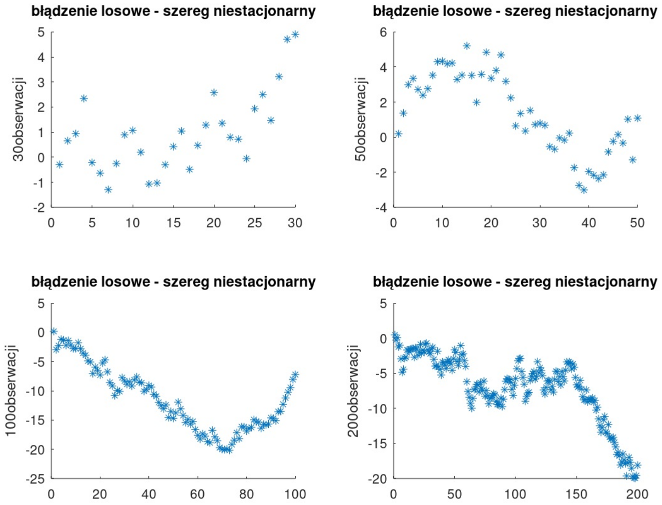
$$X_t = X_{t-1} + \epsilon_t$$

is nonstationary, while the time series

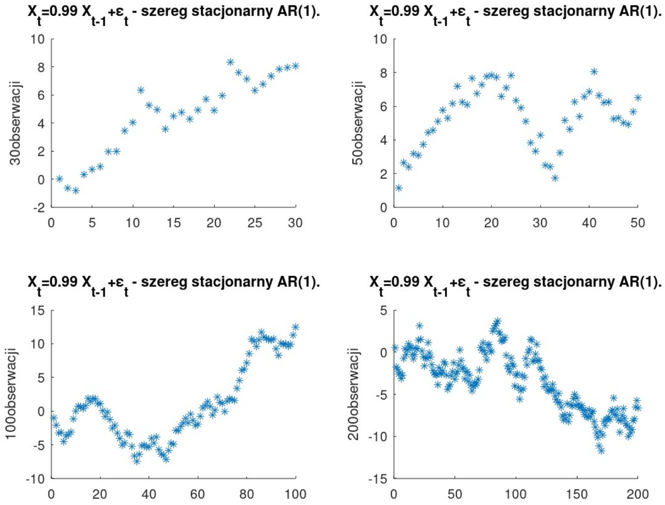
$$X_t = 0.99X_{t-1} + \epsilon_t$$

is stationary. The trajectories are both similar.

The trajectories of random walking $\sigma_t^2 \to \infty$ gdy $t \to \infty$.



The trajectories of time series $X_t = 0.99X_{t-1} + \epsilon_t$.



The diagnosis of stationarity by the trajectory.

The diagnosis of stationarity by the trajectory is difficult:

- The visual diagnosis is sufficiently difficult for μ_t and σ_t^2 whether to be constant or not;
- More difficult and even impossible is a visual verification whether $\gamma_{t,h}$ depends on t or not:
- We need to assume the stationarity first and using the statistical tool next, we need to verify whether the model fits good or not.