# Significance tests.

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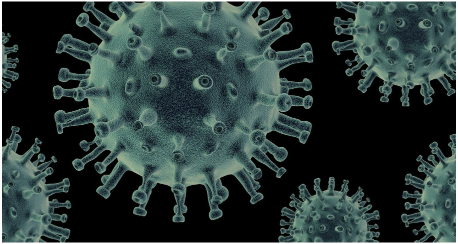


# Repetition

- The function of partial autocorrelation (PACF) - $\phi_{h,h}$ for the models AR(p) *breaks down* at the lag $p$, that is for the process in the form
$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + \epsilon_t$$
it holds
$$\phi_{p,p} \neq 0 \quad \text{and} \quad \phi_{h,h} = 0 \quad \text{for all } h = p+1, p+2, \dots, n-1.$$
- The function of autocorrelation (ACF) - $\rho_h$ for the models MA(q) *breaks down* at the lag $q$, that is for the process in the form
$$X_t = \epsilon_t - \sum_{k=1}^{q} \theta_k \epsilon_{t-k}$$
it holds
$$\rho_q \neq 0 \quad \text{and} \quad \rho_h = 0 \quad \text{for all } h = q+1, q+2, \dots, n-1.$$

# Assumption

**Assumption - the normal distribution of *white noise***

White noise $\epsilon_t$ is a collection of independent random variables whose distribution is $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 \in (0, \infty)$ is an unknown *distorting parameter*.

# Bartlett Theorem

## Bartlett Theorem

Suppose that $X_t$ is a stationary *gaussian* process (i.e. for arbitrary $v \in \mathbb{N}$ and $t_1 < t_2 < \dots t_v$, $(X_{t_1}, X_{t_2}, \dots, X_{t_v})$ has multidimensional normal distribution. Then the sampling variance $\hat{\rho}_h$ (based on the realization of $n$ sampling) of autocorrelation has the following formula

$$Var(\hat{\rho}_h) \approx \frac{1}{n} \sum_{k=-\infty}^{\infty} \left(\rho_h^2 + \rho_{k-h}\rho_{k+h} - 4\rho_k\rho_h\rho_{k-h} + 2\rho_h^2\rho_k^2\right).$$

# Application of Bartlett Theorem

If the process is MA(q), then $\rho_k = 0$ for $k = q+1, q+2, \ldots$. If additionally is gaussian we have the following corrolary:

## Corollary from Bartlett Theorem

For a gaussian MA(q) we have the following corollary:

$$Var(\hat{\rho}_h) \approx \frac{1}{n}\left(1 + 2\sum_{k=1}^{q} \rho_k^2\right) \quad \text{dla } h \geq q+1.$$

As a result for large enough $n$ (surely $n > 30$ or even more) we may accept the following approximation

$$Var(\hat{\rho}_h) \approx \frac{1}{n} \quad \text{for } h \geq q+1.$$

# Quenouille Theorem

**Twierdzenie Quenouille**

Suppose $X_t$ is a stationary and gaussian process of the type AR(p). Then the variance of the partial autocorrelation function can be expressed as follows:

$$Var(\hat{\phi}_{h,h}) \approx \frac{1}{n} \quad \text{for } h \geq p+1.$$

# Andersons Theorem

## Andersons Theorem

For gussian and stationary times series $X_t$ both estimators $\hat{\rho}_h$ and $\hat{\phi}_{h,h}$ (based on sufficiently large sampling $n$) have approximately normal distribution. Moreover,

$$\{\hat{\rho}_h : h = 1, 2, \ldots, n-1\}$$

and

$$\{\hat{\phi}_{h,h} : h = 1, 2, \ldots, n-1\}$$

are collections of approximately independent random variables.

# Tests of significance

Suppose that that $\epsilon_t$ has normal distribution, hence $X_t$ is gaussian.

- In order tho test whether the process $X_t$ is of $MA(q)$ we have to test sequentially the following hypothesis:

  $$H_0 : \rho_h = 0 \quad VS \quad \rho_h \neq 0$$

  for any $h = q+1, q+2, \dots, n-1$, in other words whether $\rho_h$ **are significant**;
- In order tho test whether the process $X_t$ is of $AR(q)$ we have to test sequentially the following hypothesis:

  $$H_0 : \phi_{h,h} = 0 \quad VS \quad \phi_{h,h} \neq 0$$

  for $h = p+1, p+2, \dots, n-1$, in other words whether $\phi_{h,h}$ **are significant**;

# Test statistics and their asymptotic distributions

From Bartlett and Quenouille Theorem we may find test statistics whose distributions have asymptotic normal distribution.

- For the tests of significance of $\rho_h$, the test statistic has the following formula:

$$R_h = \frac{\hat{\rho}_h}{\frac{1}{\sqrt{n}}} = \sqrt{n}\hat{\rho}_h;$$

- For the tests of significance of $\phi_{h,h}$, the test statistic has the following formula:

$$S_h = \frac{\hat{\phi}^{YW}_{h,h}}{\frac{1}{\sqrt{n}}} = \sqrt{n}\hat{\phi}^{YW}_{h,h}.$$

# Test statistics and their asymptotic distributions

- For the tests of significance of $\rho_h$: when the hypothesis $\rho_h = 0$ is truth, then

  $$R_h = \sqrt{n}\hat{\rho}_h \sim \mathcal{N}(0,1);$$

  **In other words $R_h$ has normal distribution;**

- For the tests of significance of $\phi_{h,h}$: when the hypothesis $\phi_{h,h} = 0$ is truth, then

  $$S_h = \sqrt{n}\hat{\phi}_{h,h}^{YW} \sim \mathcal{N}(0,1);$$

  **In other words $S_h$ has normal distribution;**

# Critical area for tests of significance

For given significance level $\kappa \in (0,1)$

- we reject the null hypothesis $\rho_h = 0$ if

$$|R_h| = \sqrt{n}|\hat{\rho}_h| > Q_{1-\frac{\kappa}{2}},$$

  where $Q_{1-\frac{\kappa}{2}}$ - the quantile of the standard normal distribution;

- we reject the null hypothesis $\phi_{h,h} = 0$ if

$$|S_h| = \sqrt{n}|\hat{\phi}_{h,h}^{YW}| > Q_{1-\frac{\kappa}{2}},$$

  where $Q_{1-\frac{\kappa}{2}}$ - the quantile of the standard normal distribution;

# Confidence area

In other words, for given significance level $\kappa \in (0,1)$ we can say

- We accept the hypothesis $\rho_h = 0$ if
$$-Q_{1-\frac{\kappa}{2}}\frac{1}{\sqrt{n}} \le \hat{\rho}_h \le Q_{1-\frac{\kappa}{2}}\frac{1}{\sqrt{n}},$$
- We accept the hypothesis $\phi_{h,h} = 0$ if
$$-Q_{1-\frac{\kappa}{2}}\frac{1}{\sqrt{n}} \le \hat{\phi}_{h,h}^{YW} \le Q_{1-\frac{\kappa}{2}}\frac{1}{\sqrt{n}}.$$

# Critical area for tests of significance

The most frequent significance levels are $\kappa = 0.01$, $\kappa = 0.05$ and $\kappa = 0.1$.

| $\kappa$ | $Q_{1-\frac{\kappa}{2}}$ | Critical area |
|---|---|---|
| 0.01 | 2.5758 | $-\frac{2.5758}{\sqrt{n}} \leq \hat{\rho}_h(\text{ or } \hat{\phi}^{YW}_{h,h}) \leq \frac{2.5758}{\sqrt{n}}$ |
| 0.05 | 1.96 | $-\frac{1.96}{\sqrt{n}} \leq \hat{\rho}_h(\text{ or } \hat{\phi}^{YW}_{h,h}) \leq \frac{1.96}{\sqrt{n}}$ |
| 0.1 | 1.6449 | $-\frac{1.6449}{\sqrt{n}} \leq \hat{\rho}_h(\text{ or } \hat{\phi}^{YW}_{h,h}) \leq \frac{1.6449}{\sqrt{n}}$ |

# Tests of significance - interpretation

Based on the presentation 17.11.2020 and todays:

- We accept the hypothesis $X_t$ to be MA(q) provided $\kappa$, if $|\rho_q|$ is much greater than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$ and
  - for $h > q$, the coefficients $|\hat{\rho}_h|$ are mostly strictly lower than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$;
  - we allow $\kappa * 100\%$ from the set $\{\hat{\rho}_{q+1}, \hat{\rho}_{q+2}, \ldots, \hat{\rho}_{n-1}\}$ to be *slightly* greater than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$.
- We accept the hypothesis $X_t$ to be AR(p) provided $\kappa$, if $|\hat{\phi}^{YW}_{p,p}|$ is much greater than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$ and
  - for $h > p$, the coefficients $|\hat{\phi}^{YW}_{h,h}|$ are mostly strictly lower than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$;
  - we allow $\kappa * 100\%$ from the set $\{\hat{\phi}^{YW}_{p+1,p+1}, \hat{\phi}^{YW}_{p+2,p+2}, \ldots, \hat{\rho}^{YW}_{n-1,n-1}\}$ to be *slightly* greater than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$.

# Tests of significance - interpretation

We accept the hypothesis $X_t$ to be *white noise* provided $\kappa$ if is MA(0) and AR(0) at the same time:

- We accept the hypothesis $X_t$ to be MA(0)
  - for $h \geq 1$, all the coefficients $|\hat{\rho}_h|$ are mostly strictly lower than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$;
  - we allow $\kappa * 100\%$ from the set $\{\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_{n-1}\}$ to be *slightly* greater than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$.
- We accept the hypothesis $X_t$ to be AR(0)
  - for $h \geq 1$, all the coefficients $|\hat{\phi}^{YW}_{h,h}|$ are mostly strictly lower than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$;
  - we allow $\kappa * 100\%$ from the set $\{\hat{\phi}^{YW}_{1,1}, \hat{\phi}^{YW}_{2,2}, \ldots, \hat{\phi}^{YW}_{n-1,n-1}\}$ to be *slightly* greater than $Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$.

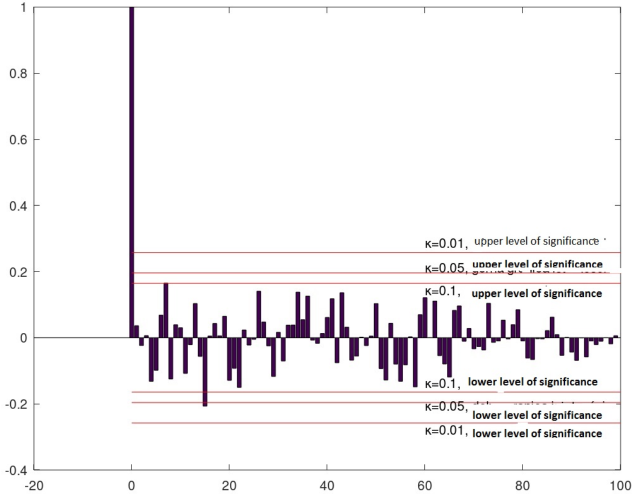
# Illustration of significance tests

## Example

We consider time series $X_t$ for $t = 1, 2, ..., n$ and we presume $X_t$ is MA(q) with unknown $q$. In this purpose, we have already computed $\hat{\rho}_h$ for $h = 1, 2, ..., n-1$, and in the plot on the next slide we have already reported:

- The high of the bar plot at $h$ is the value of $\hat{\rho}_h$, where $h$ varies from $h = 0, 1, 2, ..., n-1$;
- The red lines with corresponding $\kappa = 0.01$, $\kappa = 0.05$ and $\kappa = 0.1$ corresponds the level of significance $+/- Q_{1-\frac{\kappa}{2}} \frac{1}{\sqrt{n}}$ for corresponding $\kappa$.

# Tests of significance on levels 0.01, 0.05 i 0.1



# Tests of significance - interpretation

From the plot we can conclude

- $\hat{\rho}_7$ is inside the insignificance interval (i.e. between lower and upper level of significance), but close to the limit of significance provided that the significance levels are $\kappa = 0.1$, but $\hat{\rho}_7$ is not significant on other levels of significance;
- $\hat{\rho}_{15}$ is outside the insignificance interval, if we test the hypothesis on significance levels $\kappa = 0.01$, but inside this interval on other significance levels;
- All other coefficients $\rho_h$ are inside insignificance interval on all three significance levels.

# Tests of significance - interpretation

Summary of results in the following table.

| level $\kappa$ | $H_0 : \rho_7 = 0$ | $\rho_{15} = 0$ | $H_0 : \rho_h = 0, h \neq 7, h \neq 15$ |
|---|---|---|---|
| 0.01 | accept | accept | accept |
| 0.05 | accept | reject | accept |
| 0.1 | reject | reject | accept |

# Tests of significance - interpretation

Based on this analysis we have tho following conclusions:

- We can accept the hypothesis that $X_t$ is a white noise (MA(0)) provided $\kappa = 0.01$, since we accept all hypothesis $\hat{\rho}_h = 0$;
- We can accept the hypothesis that $X_t$ is a white noise provided $\kappa = 0.05$, since in fact $\hat{\rho}_{15}$ a unique value outside the insignificance interval, but it is close to the border line;
- We can accept the hypothesis that $X_t$ is MA(15) provided $\kappa = 0.1$ since $\hat{\rho}_{15}$ is inside the insignificance level and its moduli if fare from the line, but all other $\rho_h$ are inside insignificance interval.

## Tests of significance

**Remark**

If we considered the time series $X_t$ for $t = 1, 2, ..., n$ and if we presumed that $X_t$ are AR(q) with unknown $p$ then we could proceed similarly as in the last example provided that we compute $\hat{\phi}_{h,h}^{YW}$ instead of $\hat{\rho}_h$.