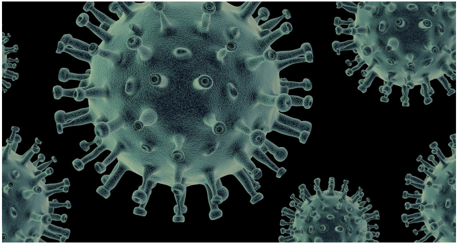
Differencing operator, integration, and ARIMA models.

15 grudnia 2020



Diffencing operator

Useful operators

For any time series X_t,

- the *shift-backward operator* is

$$BX_t = X_{t-1}$$

- the *differencing operator* is (on see 30.11.2020 slide 6 is called *differentiation*)

$$\nabla X_t = X_t - X_{t-1} = (\mathbf{I} - B)X_t;$$

Diffencing operator

Useful operators - continue

For any time series X_t,

- the *m* times composition of ∇ is

$$\nabla^m X_t = (\mathbf{I} - B)^m X_t = \underbrace{\left(\binom{m}{0}\mathbf{I} + \binom{m}{1}B + \binom{m}{2}B^2 + \ldots + \binom{m}{m}B^m\right)}_{\text{binomial theorem}} X_t$$

- the *differencing at lag m* is

$$\nabla_m X_t = X_t - X_{t-m} = (\mathbf{I} - B^m)X_t.$$

Nonstationary processes

Let ϵ_t be the white noise:

Examples

The following processes are nonstationary:

- Random walk:

$$X_t = X_{t-1} + \epsilon_t$$

- Any ARMA(p,q) processes in the polynomial form

$$w(B)X_t = \pi(B)\epsilon_t$$

  whose characteristic polynomial w has at least one root inside or on the boundary of the unit circle on the complex plane (see lecture 30.11.2020, slide 15);

Nonstationary processes

Examples - continue

The processes

$$X_t = \underbrace{m_t}_{\text{trend}} + \underbrace{\epsilon_t}_{\text{residuals}},$$

with **monotone or piecewise monotone trend** (unless trivial cases) are nonstationary:

- with **linear trend**: $m_t = a + b * t,\ a, b \in \mathbb{R}$;
- with **polynomial trend**: $m_t = a_0 + a_1 * t + \ldots + a_k t^k$, $a_0, a_1, a_2, \ldots, k \in \mathbb{R}$;
- with **exponential trend**: $m_t = a * b^t,\ a, b \in \mathbb{R}$;
- with **logarithmic trend**: $m_t = a \ln(t) + b,\ a, b \in \mathbb{R}$;

Nonstationary processes

Examples - continue

The processes

$$X_t = \underbrace{s_t}_{\text{seasonality}} + \underbrace{\epsilon_t}_{\text{residuals}},$$

with **seasonality** s_t which is a *periodic* function are nonstationary:

- with **sin waves**: $s_t = A * sin(\omega t + \varphi)$, $A, \omega, \varphi \in \mathbb{R}$;
- some combinations of **sin waves** (wave interference): $s_t = \sum_{j=1}^{k} A_j * sin(\omega_j t + \varphi_j)$, $A_j, \omega_j, a_2, \varphi_j$ for $j = 1, 2, \ldots, k$;
- s_t can be written as a T - **periodic sequence**

$$(s_1, s_2, \ldots, s_T, s_1, s_2, \ldots, s_T, s_1, s_2, \ldots, s_T, s_1, s_2, \ldots).$$

Nonstationary processes

Examples - continue

The classical decomposition model

$$X_t = \underbrace{m_t}_{\text{trend}} + \underbrace{s_t}_{\text{seasonality}} + \underbrace{Z_t}_{\text{residuals}} .$$

Here:

- m_t is a **(piecewise) monotone trend**;
- s_t is a **seasonality**:
  - s_t has a period T:
    $$s_t = s_{t+T} \quad \text{for all } t \in \mathbb{N};$$
  - the sum vanishes within the period:
    $$s_1 + s_2 + \ldots + s_T = 0.$$
- Z_t is a stationary process of residuals such that $EZ_t = 0$ for $t = 1, 2, \ldots, n$: Z_t can be a white noise, MA(q), or a stationary version of ARMA(p,q) for example.

Eliminations of trend

Elimination of trend - examples

Due to the differencing operators we may eliminate nonstationarities of some nonstationary time series:

- Random walk is not stationary, but:

  $$X_t = X_{t-1} + \epsilon_t \Leftrightarrow \nabla X_t = \epsilon_t;$$

  hence ∇X_t is stationary;
- The model with linear trend $X_t = a + bt + \epsilon_t$ is nonstationary. But

  $$\left.\begin{array}{rcl} X_t &=& a + bt + \epsilon_t \\ X_{t-1} &=& a + b(t-1) + \epsilon_{t-1} \end{array}\right\} \text{by subtracting sides} \quad \underbrace{\nabla X_t}_{X_t - X_{t-1}} = b + \epsilon_t - \epsilon_{t-1}.$$

  Hence ∇X_t is MA(1), hence is stationary.

Eliminations of trend

Eliminations of trend - examples

Let

$$X_t = a_0 + a_1 t + a_2 t^2 + \epsilon_t .$$

We have

$$\left.\begin{array}{rcc} X_t & = & a_0 + a_1 t + a_2 t^2 + \epsilon_t \\ X_{t-1} & = & a_0 + a_1 (t-1) + a_2 (t-1)^2 + \epsilon_{t-1} \end{array}\right\} \text{ by subtracting sides}$$

we still have a nonstationary time series

$$\nabla X_t = \underbrace{-a_2 + (2a_2 + a_1) t}_{\text{linear trend}} + \epsilon_t - \epsilon_{t-1} .$$

Applying ∇X_t again, we obtain a stationary process in the following form:

$$\nabla^2 X_t = \underbrace{2a_2 + a_1}_{\text{a constant}} + \underbrace{\epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}}_{\text{MA(2)}} .$$

Eliminations of seasonality

Eliminations of seasonality - examples

Consider the time series

$$X_t = s_t + \epsilon_t.$$

where s_t is the seasonality of period T. Then, We have

$$\left.\begin{array}{rcc} X_t & = & s_t + \epsilon_t \\ X_{t-T} & = & s_{t-T} + \epsilon_{t-T} \end{array}\right\} \text{ by subtracting sides}$$

we still have a stationary time series

$$\nabla_T X_t = \underbrace{s_t - s_{t-T}}_{=0 \text{ by } T \text{ - periodicity}} + \epsilon_t - \epsilon_{t-T}.$$

Hence the new series $\nabla_T X_t$ is MA(T) with

$$\underbrace{\nabla_T X_t}_{\text{new time series}} = \underbrace{\epsilon_t - 0\epsilon_{t-1} - 0 * \epsilon_{t-2} - \ldots - 0\epsilon_{t-(T-1)} - \epsilon_{t-T}}_{MA(T)}$$

Elimination of trend and seasonality

Fact

Let X_t be the stationary process such that $EX_t = \mu$ and $\gamma(h) = Cov(X_t, X_{t+h})$. Then ∇X_t is stationary with 0 expectation.

Proof

We have:

$$E\nabla X_t = E(X_t - X_{t-1}) = EX_t - EX_{t-1} = \mu - \mu = 0,$$

hence does not depend on t

$$\begin{aligned} Cov(\nabla X_t, \nabla X_{t+h})) &= Cov(X_t - X_{t-1}, X_{t+h} - X_{t-1+h}) \\ &= Cov(X_t, X_{t+h}) - Cov(X_{t-1}, X_{t+h}) \\ &\quad - Cov(X_t, X_{t-1+h}) + Cov(X_{t-1}, X_{t-1+h}) \\ &= \underbrace{2\gamma(h) - \gamma(h+1) - \gamma(h-1)}_{\text{does not depend on } h}. \end{aligned}$$

As a result, ∇X_t is stationary.

Elimination of trend and seasonality

Fact

Let X_t be the stationary process. Then for any $m \in \mathbb{N}$, $\nabla^m X_t$ is stationary with zero expectation.

Elimination of trend and seasonality

Proof

Let X_t be a stationary process. We show that for any m, $E\nabla^m X_t = 0$ and that the autocovariance of $\nabla^m X_t$ does not depend on t.

- From the previous fact, ∇X_t is stationary and $E\nabla X_t = 0$, hence for $m = 1$ this thesis is true;
- Suppose that $\nabla^m X_t$ is stationary and $E\nabla^m X_t = 0$ for some m;
- Then

  $$\nabla^{m+1} X_t = \nabla(\nabla^m X_t),$$

  hence $E\nabla^{m+1} X_t = 0$ and $\nabla^{m+1} X_t$ is stationary by induction hypothesis ($\nabla^m X_t$ is stationary) and the previous fact. As a result, for any m, $\nabla^m X_t$ is stationary with zero expectation.

Elimination of trend and seasonality

Fact

Let X_t be the stationary process. Then for any $m \in \mathbb{N}$, $\nabla_m X_t$ is stationary with zero expectation.

Proof

We have:

$$E\nabla_m X_t = E(X_t - X_{t-m}) = EX_t - EX_{t-m} = \mu - \mu = 0,$$

hence does not depend on t

$$\begin{aligned} Cov(\nabla_m X_t, \nabla_m X_{t+h})) &= Cov(X_t - X_{t-m}, X_{t+h} - X_{t+h-m}) \\ &= Cov(X_t, X_{t+h}) - Cov(X_{t-m}, X_{t+h}) \\ &\quad - Cov(X_t, X_{t+h-m}) + Cov(X_{t-m}, X_{t+h-m}) \\ &= \underbrace{2\gamma(h) - \gamma(h+m) - \gamma(h-m)}_{\text{does not depend on } h}. \end{aligned}$$

As a result, $\nabla_m X_t$ is stationary.

Elimination of trend and seasonality

Consider the classical decomposition model as follows:

$$X_t = m_t + s_t + Z_t$$

where

- m_t - polynomial trend;
- s_t - the seasonality with period T and

$$s_1 + s_2 + \ldots + s_T = 0;$$

- Z_t is the stationary time series with $EZ_t = 0$.

Elimination of trend and seasonality

- If m_t is a polynomial of k degree then

$$\nabla^k m_t = constant$$

- As a result,

$$\begin{aligned} \nabla^k X_t &= \nabla^k (m_t + s_t + Z_t) \\ &= \underbrace{\nabla^k m_t}_{=\text{constant}} + \underbrace{\nabla^k s_t}_{\text{new seasonality}} + \underbrace{\nabla^m Z_t}_{\text{stationary process}} \end{aligned}$$

- Hence $\nabla^k X_t$ has the following form:

$$\nabla^k X_t = \text{constant} + \text{seasonality} + \text{stationary time series};$$

- **k times diffrentiation eliminates the polynomial trend with degree k.**

Elimination of trend and seasonality

- Suppose m_t is k degree polynomial and s_t is a T period seasonality;
- Then

$$\nabla_T s_t = s_t - s_{t-T} = 0 \quad \text{and} \quad \nabla_T m_t = \underbrace{m_t - m_{t-T}}_{\text{polynomial k-1 degree}} \quad ;$$

- As a result,

$$\begin{aligned} \nabla_T X_t &= \nabla_T (m_t + s_t + Z_t) \\ &= \underbrace{\nabla_T m_t}_{=\text{polynomial of degree } k-1} + \underbrace{\nabla_T s_t}_{=0} + \underbrace{\nabla^m Z_t}_{\text{stationary process}} \end{aligned}$$

- Hence $\nabla_T X_t$ has the following form:

$$\nabla_T X_t = \text{constant} + \text{seasonality} + \text{stationary time series};$$

- **Differencing at lag T eliminates the T period seasonality and reduces degree of the polynomial trend by 1.**

Integration

Integration of time series

The nonstationary process X_t is **integrated of order** $k \geq 1$ if

- $\nabla^k X_t$ is stationary,
- but $\nabla^{k-1} X_t$ is not stationary (in case of 0 differencing $\nabla^0 X_t = X_t$).

Integration

Examples

Integrated processes:

- The process of classical decomposition

$$X_t = a_0 + a_1 t + \ldots + a_k t^k + Z_t$$

  where s_t is a seasonality and Z_t is a stationary process is integrated of order k.
- The standard random walking $X_t = X_{t-1} + \epsilon_t$ is nonstationary, but ∇X_t is stationary. Hence X_t is integrated of order 1.

Stationarity and Integration of ARMA(p,q)

Stationarity of ARMA(p,q)

Consider the process ARMA(p,q) in the polynomial form

$$w(B)X_t = \pi(B)\epsilon_t$$

If all roots of the characteristic polynomial $w(\lambda)$ lie outside the unit circle of the complex plane, then ARMA(p,q) is **stationary**.

Stationarity and Integration of ARMA(p,q)

Integration of ARMA(p,q)

Consider the ARMA(p,q) process

$$w(B)X_t = \pi(B)\epsilon_t$$

in which 1 is k times root, but all other roots $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m$ lie outside the unit circle of the compex plane. Then

$$w(B) = (1-B)^{\boldsymbol{k}} \left(\mathbf{1} - \frac{1}{\lambda_{\boldsymbol{k+1}}} B\right) \left(\mathbf{1} - \frac{1}{\lambda_{\boldsymbol{k+2}}} B\right) \ldots \left(\mathbf{1} - \frac{1}{\lambda_{\boldsymbol{m}}} B\right).$$

Then ARMA(p,q) is **integrated of order** k.

Stationarity and Integration of ARMA(p,q)

Integration of ARMA(p,q)-continue

Indeed,

- we can express

$$\underbrace{\left(I - \frac{1}{\lambda_{k+1}}B\right)\left(I - \frac{1}{\lambda_{k+2}}B\right)\dots\left(I - \frac{1}{\lambda_m}B\right)}_{\text{polynomial of stationary operator}}(I-B)^k X_t = \pi(B)\epsilon_t,$$

  hence $(I - B)^k X_t$ is **stationary**.

- but we can alternatively express

$$\underbrace{\left(I - \frac{1}{\lambda_{k+1}}B\right)\left(I - \frac{1}{\lambda_{k+2}}B\right)\dots\left(I - \frac{1}{\lambda_m}B\right)(I-B)}_{\text{polynomial of nonstationary operator}}(I-B)^{k-1} X_t = \pi(B)\epsilon_t,$$

  hence $(I - B)^{k-1} X_t$ is **not stationary**.

ARIMA

ARIMA(p,d,q)

The time series X_t is in ARIMA(p,d,q) if the differentiation $\nabla^d X_t$ is ARMA(p,q).

Remark

Observe that ARIMA(p,d,q) is in fact the special case of ARMA(p+d,q) with higher lags. For example, the process ARIMA(1,1,1) can be written as

$$\nabla X_t = \phi \nabla X_{t-1} + \epsilon_t - \epsilon_{t-1}.$$

Hence

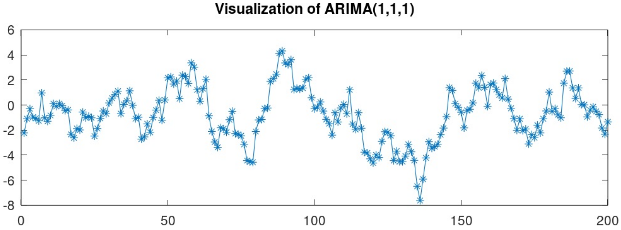
$$X_t - X_{t-1} = \phi(X_{t-1} - X_{t-2}) + \epsilon_t - \epsilon_{t-1} \Leftrightarrow \underbrace{X_t = (1+\phi)X_{t-1} - \phi X_{t-2} + \epsilon_t - \epsilon_{t-1}}_{ARMA(2,1)}.$$

Illustration of trajectory ARIMA(p,d,q)

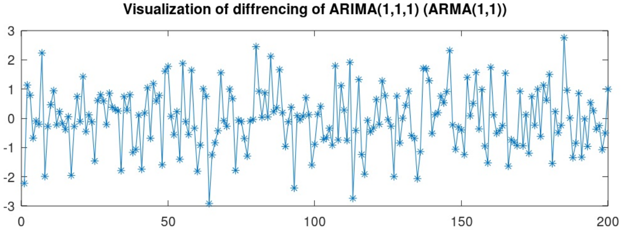
Consider the ARIMA(p,d,q) that is

- Integrated of order d;
- That is, $\nabla^d X_t$ is a stationary ARMA(p,q);
- The trajectory X_t should be a plot that is ones increasing, next decreasing, next turn to be increasing and so on.
- But the differentiation should irregularly oscillate around 0

Illustration of trajectory ARIMA(p,d,q)



Visualization of ARIMA(1,1,1)



Visualization of diffrencing of ARIMA(1,1,1) (ARMA(1,1))

Illustration of trajectory ARIMA(p,d,q)

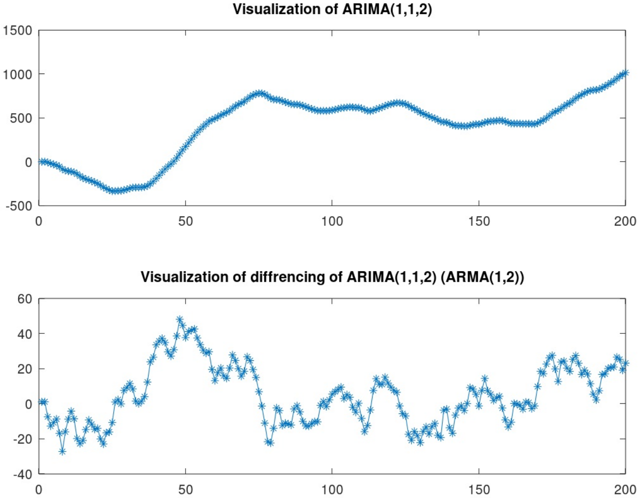
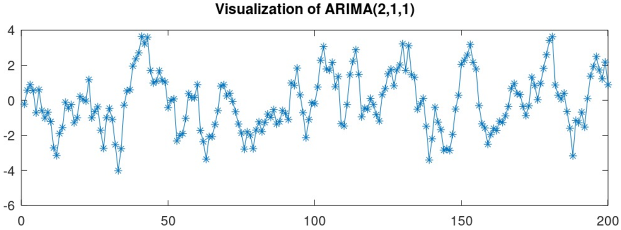


Illustration of trajectory of ARIMA(p,d,q)



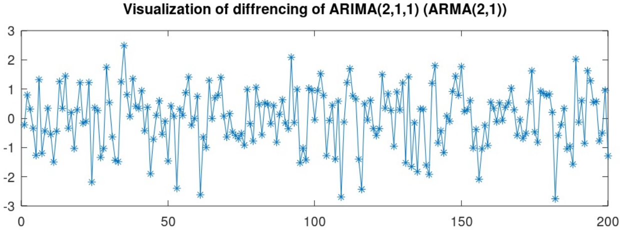


Illustration of ACF of ARIMA(p,d,q)

If the time series is ARIMA(p,d,q) such that

- integrated of order d;
- or equivalently, $\nabla^d X_t$ is the stationary process ARMA(p,q).

The bar plot of ACF:

- the ACF of X_t should tend to 0 as a (rather slow) suppressed wave;
- the ACF of $\nabla^d X_t$ should tend to 0 as a suppressed wave, faster than the ACF of X_t;

Illustration of ACF of ARIMA(p,d,q)

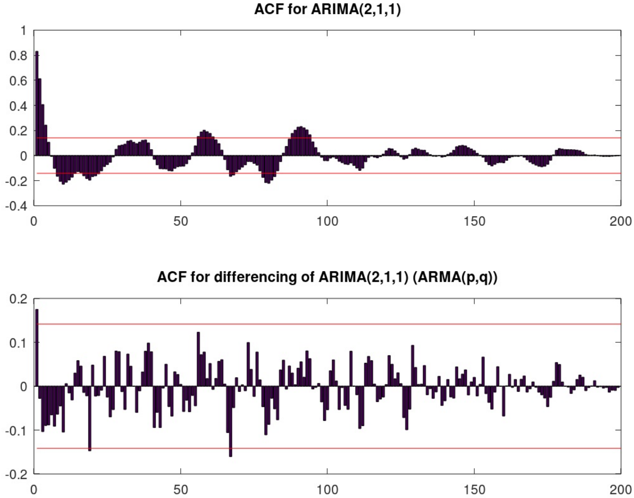


Illustration of PACF of ARIMA(p,d,q)

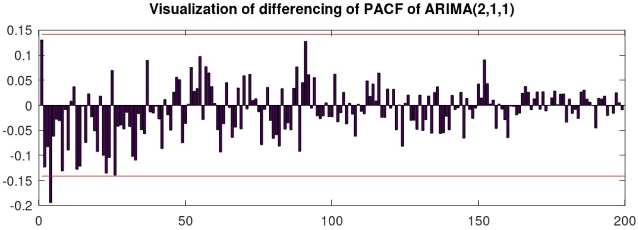
If the time series is ARIMA(p,d,q) such that

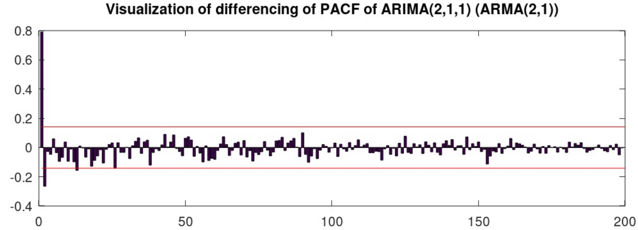
- integrated of order d;
- or equivalently, $\nabla^d X_t$ is a stationary ARMA(p,q) process.

The bar plot of PACF:

- the PACF of X_t should tend to 0 as a (rather slow) suppressed wave, or bread down very late;
- the PACF of $\nabla^d X_t$ should tend to 0 as a suppressed wave, faster than the PACF of X_t, or break down faster than X_t;

Illustration of PACF of ARIMA(p,d,q)





Identification of ARIMA(p,d,q)

The following table summarizes the identification of ARIMA(p,d,q) by

- the trajectory of X_t and $\nabla^d X_t$;
- the barplot of ACF of X_t and $\nabla^d X_t$;
- the barplot of PACF of X_t and $\nabla^d X_t$;

	X_t	$\nabla^d X_t$
trajectory	pieswise monotone	oscillating around 0
ACF	tends slowly to 0	tends faster to 0
PACF	tends slowly to 0	tends fast to 0 or breaks down

Correlogram for classical decomposition

Correlogram of the classical decomosition model

The classical decomposition model:

$$X_t = m_t + s_t + Z_t$$

has the same ACF and PACF as the stationary residuals Z_t: if $\gamma(h) = Cov(Z_t, Z_{t+h})$ then,

$$Cov(X_{\boldsymbol{t}}, X_{\boldsymbol{t+h}}) = Cov(m_{\boldsymbol{t}} + s_{\boldsymbol{t}} + Z_{\boldsymbol{t}}, m_{\boldsymbol{t+h}} + s_{\boldsymbol{t+h}} + Z_{\boldsymbol{t+h}}) = Cov(Z_{\boldsymbol{t}}, Z_{\boldsymbol{t+h}}) = \gamma(h).$$

But this model is not stationary since

$$EX_t = m_t + s_t$$

which depends on t.