GARCH models.

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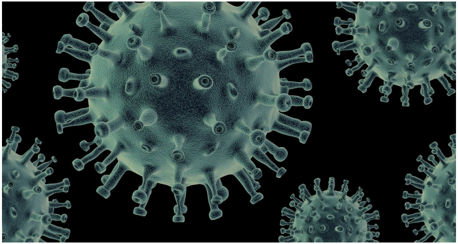

Linear models

All of the following models of time series X_t

- classical decomposition model;
- AR(p), MA(q), ARMA(p,q);
- white noise

are linear transformations of ϵ_t.

Linear models

For example,

- in case of MA(q)

$$X_t = \epsilon_t - \theta_1 \epsilon_{t-1} - \ldots - \theta_q \epsilon_{t-q}$$

  is a linear transformation of $(\epsilon_t, \epsilon_{t-1}, ..., \epsilon_{t-q})$

- in case of *stationary* AR(p), the model can be reformulated as $MA(\infty)$ in the following form

$$X_t = \epsilon_t - \sum_{k=1}^{\infty} \theta_k \epsilon_{t-k}$$

  hence a linear transformation of $\epsilon_t, \epsilon_{t-1}, \ldots$;

- Similarly we conclude that the *stationary* process ARMA(p,q) is linear transformation as well;
- The white noise is identity of itself, hence linear.

Remarks on sigma fields and the conditional probability

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $\mathcal{F} \subset \mathcal{B}$ and suppose that $\mathcal{F}$ is a sigma-field. Let X be a random variable (measurable in $\mathcal{B}$). Then $E(X|\mathcal{F})$ is a **conditional expecation of X under $\mathcal{F}$** if:

- $E(X|\mathcal{F})$ is an $\mathcal{F}$- measurable function;
- For any $A \in \mathcal{F}$

$$\int_A E(X|\mathcal{F})dP = \int_A XdP.$$

In particular, since $\Omega \in \mathcal{F}$ (the full set belongs to any σ field).

$$E(E(X|\mathcal{F})) = \int_\Omega E(X|\mathcal{F})dP = \int_\Omega XdP = E(X).$$

Remarks on sigma fields and the conditional probability

Fact

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $\mathcal{F} \subset \mathcal{B}$ and suppose that $\mathcal{F}$ is a sigma-field. If X and Y are both random variables such that $EX^2 < \infty$ and $EY^2 < \infty$ and Y is $\mathcal{F}$- measurable, then

$$E(XY|\mathcal{F}) = YE(X|\mathcal{F}) \quad P - a.e.$$

Definition

Let $(X_t)_{t \in T}$ be a family of random variables. We define $\sigma(\{X_t : t \in T\})$ as follows:

$$\sigma(\{X_t : t \in T\}) = \sigma\left(\{X^{-1}(A) : A \in \mathcal{B}\}\right).$$

Definition

We say that the sequence of random variables X_t is *homoscedastic* if the variance $Var(X_t)$ is a constant (time invariant) value. Otherwise, if $Var(X_t)$ does depend on t, the sequence X_t is *heteroscedastic*.

Example

The white noise ϵ_t is homoscedastic since $Var(\epsilon_t) = \sigma^2$ is constant. Similarly any stationary time series is homoscedastic. But the standard white random walk in the form

$$X_t = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_t$$

is heteroscedastic since $Var(\epsilon_t) = t\sigma^2$ (depends on t).

Autoregressive Conditional Heteroscedasticity ARCH

Notation

The notation $\epsilon_t \sim \mathit{IID}\, N(0,1)$ means that ϵ_t is a sequence of independent identically distributed random variables, and any of ϵ_t has the standard normal distribution $\mathcal{N}(0,1)$.

ARCH(p)

The time series X_t is ARCH(p) if has the form

$$X_t = \epsilon_t \sqrt{\alpha_0 + \sum_{k=1}^{p} \alpha_k X_{t-k}^2},$$

where $\epsilon_t \sim \mathit{IID}\, N(0,1)$ and $\alpha_0, \alpha_1, \ldots, \alpha_p$ are unknown parameters. The short ARCH means Autoregressive Conditional Heteroscedasticity.

ARCH(p) - properties

Let $\mathcal{F}_t := \sigma(\epsilon_t, \epsilon_{t-1}, \ldots)$ and let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{Z}}$ be a *filtration*.

- The ARCH(p) process X_t is $\mathcal{F}_t$ - measurable. In other words, the sequence X_t is adapted to the filtration $\mathcal{F}_t$;
- The processes AR(p), MA(q), ARMA(p,q) and the classical decomposition models are adapted to $\mathcal{F}$ as well;
- The processes AR(p), MA(q), ARMA(p,q) and the classical decomposition models, are *linear* transformations of $(\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots;$
- The process ARCH(p) is a *nonlinear* transformation of $(\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots);$

The model ARCH(1)

The simplest ARCH(1) model has the form:

$$X_t = \epsilon_t \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}$$

with $\alpha_1 \in (0,1)$. After squaring both sides we have

$$X_t^2 = \epsilon_t^2 (\alpha_0 + \alpha_1 X_{t-1}^2).$$

We have the following fact.

The model ARCH(1)

Fact

The model ARCH(1) with $\alpha_1 \in (0, 1)$ satisfies the following:

- vanishing expectation: $E(X_t) = 0$;
- homoscedasticity: $Var(X_t) = EX_t^2 = \frac{\alpha_0}{1-\alpha_1}$;
- pairwise uncorrelated: $Cov(X_t, X_{t+h}) = 0$ for any $h \neq 0$.

In particalar, X_t is a stationary time series.

The model ARCH(1) (vanishing expectation)

We show $EX_t = 0$ for all t. Indeed,

$$\begin{aligned} EX_t &= E\left(\epsilon_t \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}\right) \\ &= E\left(E\left(\epsilon_t \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2} | \mathcal{F}_{t-1}\right)\right) \\ &= E\left(\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2} E\left(\epsilon_t | \mathcal{F}_{t-1}\right)\right) \end{aligned}$$

The last equality follows that X_{t-1} is $\mathcal{F}_{t-1}$- measurable. Since ϵ_t is independent on $\mathcal{F}_{t-1}$ we have

$$E(\epsilon_t | \mathcal{F}_{t-1}) = E(\epsilon_t).$$

Therefore, since $E(\epsilon_t) = 0$ hence

$$EX_t = E\left(E(\epsilon_t)\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}\right) = 0.$$

The model ARCH(1) (homoscedasticity)

We are going to prove this fact. Now we expand the expression of X_t^2 to arbitrary n

$$
\begin{aligned}
X_t^2 &= \epsilon_t^2 \alpha_0 + \epsilon_t^2 \alpha_1 X_{t-1}^2 \\
&= \epsilon_t^2 \alpha_0 + \alpha_1 \alpha_0 \epsilon_t^2 \epsilon_{t-1}^2 + \alpha_1^2 X_{t-2}^2 \epsilon_t^2 \epsilon_{t-1}^2 \\
&= \alpha_0 \sum_{k=0}^{n} \alpha_1^k \epsilon_t^2 \epsilon_{t-1}^2 \dots \epsilon_{t-k}^2 + \alpha_1^{n+1} X_{t-n-1} \epsilon_t^2 \epsilon_{t-1}^2 \dots \epsilon_{t-n}^2.
\end{aligned}
$$

The model ARCH(1) (homoscedasticity)

Taking the limit in $n \to \infty$ and noting that $\alpha_1^n \to 0$

$$X_t^2 = \alpha_0 \sum_{k=0}^{\infty} \alpha_1^k \epsilon_t^2 \epsilon_{t-1}^2 \dots \epsilon_{t-k}^2,$$

hence

$$\begin{aligned} EX_t^2 &= E\left(\alpha_0 \sum_{k=0}^{\infty} \alpha_1^k \epsilon_t^2 \epsilon_{t-1}^2 \dots \epsilon_{t-k}^2\right) \\ &= \alpha_0 E\left(\sum_{k=0}^{\infty} \alpha_1^k \epsilon_t^2 \epsilon_{t-1}^2 \dots \epsilon_{t-k}^2\right). \end{aligned}$$

The model ARCH(1) (homoscedasticity)

By Dominating Convergence Theorem, independence of ϵ_t and since $\epsilon_t \sim \mathcal{N}(0,1)$ we have we have

$$\begin{aligned} EX_t^2 &= \alpha_0 \sum_{k=0}^{\infty} \alpha_1^k E(\epsilon_t^2 \epsilon_{t-1}^2 \dots \epsilon_{t-k}^2) \\ &= \alpha_0 \sum_{k=0}^{\infty} \alpha_1^k E(\epsilon_t^2) E(\epsilon_{t-1}^2) \dots E(\epsilon_{t-k}^2) \\ &= \alpha_0 \sum_{k=0}^{\infty} \alpha_1^k = \frac{\alpha_0}{1-\alpha_1}, \end{aligned}$$

Since $EX_t = 0$, hence

$$Var(X_t) = EX_t^2 = \frac{\alpha_0}{1-\alpha_1},$$

hence we have the homoscedasticity.

The model ARCH(1) (pairwise uncorrelated)

We show $Cov(X_{t+h}, X_t) = 0$ for $h > 0$. Since X_t is $\mathcal{F}_{t+h-1}$ -measurable, hence

$$\begin{aligned} E(X_{t+h}X_t) &= E(E(X_{t+h}X_t|\mathcal{F}_{t+h-1})) \\ &= E(X_t E(X_{t+h}|\mathcal{F}_{t+h-1})). \end{aligned}$$

Furthermore, since X_{t+h-1} (and consequently $\sqrt{\alpha_0 + \alpha_1 X_{t+h-1}}$) is $\mathcal{F}_{t+h-1}$ measurable, hence

$$\begin{aligned} E(X_{t+h}|\mathcal{F}_{t+h-1})) &= E\left(\epsilon_{t+h}\sqrt{\alpha_0 + \alpha_1 X_{t+h-1}}|\mathcal{F}_{t+h-1}\right) \\ &= E(\epsilon_{t+h}|\mathcal{F}_{t+h-1})\sqrt{\alpha_0 + \alpha_1 X_{t+h-1}}. \end{aligned}$$

The model ARCH(1) (pairwise uncorrelated)

Since ϵ_{t+h} is idenpendent on $\mathcal{F}_{t+h-1}$ hence

$$E(\epsilon_{t+h}|\mathcal{F}_{t+h-1}) = E(\epsilon_{t+h}) = 0.$$

Consequently,

$$E(X_{t+h}|\mathcal{F}_{t+h-1})) = E(\epsilon_{t+h}|\mathcal{F}_{t+h-1})\sqrt{\alpha_0 + \alpha_1 X_{t+h-1}} = 0.$$

As a result,

$$Cov(X_{t+h}, X_t) = E(X_{t+h}X_t) = 0.$$

The proof is complete.

The model ARCH(1)

Corollary ACF for ARCH(1)

The ACF for ARCH(1) with $\alpha_1 \in (0, 1)$ behaves like a white noise.

Proof of Corollary

$$Corr(X_t, X_t) = \frac{Var(X_t)}{Var(X_t)} = \frac{\frac{\alpha_0}{1-\alpha_1}}{\frac{\alpha_0}{1-\alpha_1}} = 1.$$

and

$$Corr(X_{t+h}, X_t) = 0,$$

hence the autocorrelation is the same as that of white noise.

Remark

We have

- The distribution X_{t+1} under condition $\mathcal{F}_t$ in
$$\mathcal{N}\left(0, \alpha_0 + \sum_{k=1}^{p} \alpha_k X_{t-k}^2\right);$$
- Hence
$$Var(X_{t+1}|\mathcal{F}_t) = \alpha_0 + \sum_{k=1}^{p} \alpha_k X_{t-k}^2.$$
- As a result the variance of X_t varies in time conditionally on $\mathcal{F}_t$, hence is **conditionally heteroscedastic**.

Remark - continue

- In particaular, ARCH(1) is **conditionally heteroscedastic** since

$$Var(X_{t+1}|\mathcal{F}_t) = \alpha_0 + \alpha_1 X_{t-1}^2$$

varies conditionally on $\mathcal{F}_t$, although as we have proven X_t is *homoscedastic* ($Var(X_t) = const$).

The model GARCH(p,q)

GARCH(p,q)

The process X_t is GARCH(p,q) if has the following form

$$X_t = \epsilon_t \sqrt{h_t},$$

where h_t has the following form

$$h_t = \alpha_0 + \sum_{k=1}^{p} \alpha_k X_{t-k}^2 + \sum_{k=1}^{q} \beta_k h_{t-k}^2$$

and

$$\epsilon_t \sim IID\, \mathcal{N}(0,1).$$

In particular, ARCH(p) is GARCH(p,0).

Correlogram of GARCH(p,q)

- The correlogram (ACF, PACF) of stationary GARCH(p,q) behaves like MA(q);
- The correlogram of stationary ARCH(q) behaves like white noise;
- If corellogram ACF and PACF breaks up in large lag, the model GARCH(p,q) is sometimes more appropriate than ARMA or ARIMA;
- The data of stock exchange often fits to GARCH(p,q).

Correlogram of stationary ARCH(1) - example