Decision Analysis and Decision Theory

The aim of this lecture is an in-depth mathematical analysis of processes in which decisions have to be made and each choice involves risk.

1 Optimal Portfolio Construction

The following *primary financial instruments*, among others, are available on the market:

- **Stock** – a security representing an ownership interest in the capital of a corporation and conferring property and corporate rights.
- **Bond** – a security under which the issuer acknowledges its debt to the bondholder and undertakes to provide a specified payment or other benefit.
- **Bill of exchange** (pol. Weksel) – a security containing an unconditional promise or order to pay a specified sum of money.
- **Participation certificate** (pol. Świadectwo udziałowe) – a document confirming specified property rights associated with participation in an entity or venture.
- **Deposit** – an instrument representing a claim against a financial institution, usually involving relatively low risk.

There are also *derivative instruments*. The most important ones include:

- **Forward contract** – a non-standardized agreement entered into directly between two parties, obligating them to buy and sell a specified asset at a future date at a price agreed upon when the contract is entered into.
- **Futures contract** – a standardized exchange-traded contract obligating the parties to buy and sell a specified asset at a future date at an agreed price.

- **Option** – a contract giving its holder the right, but not the obligation, to buy (*call*) or sell (*put*) a specified asset at an agreed price, on or within a specified period.

Each financial instrument offers an *expected rate of return*, but also involves *risk*:

- A primary financial instrument may be purchased at an attractive price with the intention of selling it later at a higher price, but there is no guarantee that the expected price increase will ever occur;
- *Forward* and *futures* contracts obligate the parties to execute a transaction at an agreed price, thereby providing protection against adverse price movements, but they also create the risk that the price specified in the contract will turn out to be less favorable than the prevailing market price;
- Options also protect the holder against adverse price movements without obligating the holder to execute the transaction; however, this right requires the payment of an option premium.

1.1 The asset market

We have N assets in two periods of time 0 and 1, and the price of each asset i at 0 is S_0^i and at 1 is S_1^i.

Definition 1. *The* ***rate of return of asset*** i *from* 0 *to* 1 *is*

$$R_i := \frac{S_1^i - S_0^i}{S_0^i}.$$

The asset market is specified by the assets $\{1, 2, \ldots, N\}$, their rates of returns $(R_1, R_2, \ldots, R_N)$ and asset prices $(S_0^1, S_0^2, \ldots, S_0^N)$, and $(S_1^1, S_1^2, \ldots, S_1^N)$.

Definition 2. *The investors* ***portfolio*** *is* $x := (x_1, x_2, \ldots, x_N)$ *where each* $x_i \in \mathbb{R}$ *means an amount invested in the asset* i*.*

The negative coordinates ($x_i < 0$) at the portfolio encompasses all of the following three cases:

1. The investor has borrowed a money (for example from a bank) for buying the asset, and has to pay a debt later (with some interest rate);

2. The negativity means that short-selling happens. That means that the investor has borrowed an asset (for example from a broker) to sell on the market at time 0, and later the asset has to be returned;

3. The negativity may be provided by a long position in futures or forward contract, or by put option. The investor is obligated to buy asset i by $-x_i$ later;

We introduce standard terminology in the finance.

Definition 3. *In a portfolio $(x_1, x_2, \ldots, x_N)$, for each i, a negative value $x_i < 0$ represents a **short position** in asset i, while a positive value $x_i > 0$ represents a **long position** in asset i.*

1.2 Return portfolio

Let us recall that the asset market is specified by the assets $\{1, 2, \ldots, N\}$, their rates of returns $(R_1, R_2, \ldots, R_N)$ and asset prices $(S_0^1, S_0^2, \ldots, S_0^N)$, and $(S_1^1, S_1^2, \ldots, S_1^N)$. Suppose that the investor at time 0 has a portfolio $(x_1, x_2, \ldots, x_N)$.

Definition 4. *The **initial wealth of portfolio** is*

$$w_0 = x_1 + x_2 + \ldots + x_N.$$

*The **terminal wealth of portfolio** is*

$$w_1 = \frac{S_1^1}{S_0^1}x_1 + \frac{S_1^2}{S_0^2}x_2 + \ldots + \frac{S_1^N}{S_0^N}x_N.$$

In other words, each coordinate i of the portfolio is strengthened by the ratio between future and present price $\frac{S_1^i}{S_0^i}$. Simple calculations yield

$$w_1 = \sum_{i=1}^{N} \frac{S_1^i}{S_0^i} x_i = \sum_{i=1}^{N} \frac{S_1^i - S_0^i}{S_0^i} x_i + \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} R_i x_i + w_0. \tag{1}$$

We provide definitions of specific portfolios with initial wealth 0 or 1.

Definition 5. *The portfolio* $(x_1, x_2, \ldots, x_N)$ *is* ***normalized*** *if*

$$x_1 + x_2 + \ldots + x_N = 1.$$

The portfolio is ***self-financing*** *if*

$$x_1 + x_2 + \ldots + x_N = 0.$$

Next we provide return and net return of portfolio.

Definition 6. *For the portfolio* $x = (x_1, x_2, \ldots, x_N)$ *the* ***return of portfolio*** *is defined as*

$$w_1 - w_0.$$

If $w_0 \neq 0$ *then the* ***net return of portfolio*** *is defined as*

$$\frac{w_1 - w_0}{w_0}.$$

Using (1) the net of portfolio can be expressed as follows:

$$\frac{w_1 - w_0}{w_0} = \frac{\sum\limits_{i=1}^{N} R_i x_i}{\sum\limits_{i=1}^{N} x_i} = \sum_{i=1}^{N} R_i \left(\frac{x_i}{\sum\limits_{j=1}^{N} x_j} \right) = \sum_{i=1}^{n} R_i w_0^i,$$

where w_0^i is a proportion of initial wealth invested at asset i. In particular, for normalized portfolio return and net return coincide and

$$w_1 - w_0 = \frac{w_1 - w_0}{w_0} = \sum_{i=1}^{N} R_i x_i.$$

1.3 Markowitz model

Before we formulate formally the **Markowitz model** we recall some notations from the probability theory.

1.3.1 Standard notations

Let $(\Omega, \mathcal{B}, P)$ be a probability space and let X and Y be random variables.

- The **expected value** of random variable X is denoted as

$$EX = \int_{\Omega} X(\omega) dP(\omega);$$

- The **variance** of X is denoted as follows:

$$Var(X) = E(X - EX)^2 = EX^2 - (EX)^2;$$

- The **covariance** of random varables X and Y is denoted as follows:

$$Cov(X, Y) := E(X - EX)(Y - EY) = E(XY) - (EX)(EY);$$

In particular, it holds

$$Cov(X, X) = Var(X)$$

1.3.2 Markowitz model-description

The Markowitz model describes N assets characterized by random rates of returns

$$(R_1, R_2, \ldots, R_N)$$

such that for each i, ER_i^2 exists and is finite.

Remark 1. *Since for each i, $ER_i^2 < \infty$, hence ER_i also exists and is finite. Moreover, $Var(R_i)$ exists and is finite. Moreove, for each i and j, $Cov(R_i, R_j)$ exists and is finite.*

We provide the following notations:

- For each i, the **expected rate of return** is $m_i := ER_i$ and the vector is written as a vector $m = (m_1, m_2, \ldots, m_N)$;
- For each i and j we denote a **covariance** between expected rates of returns of i and j as follows $\sigma_{ij} = Cov(R_i, R_j)$; and the **covariance matrix** is

$$\Sigma = [\sigma_{ij}]_{i=1,\ldots,N, j=1,\ldots,N}.$$

The interpretation of Σ is crucial.

Remember: Interpretation of the coefficients in the covariance matrix Σ:

- $\sigma_{ii} = \mathrm{Var}(R_i)$ represents the variance of the rate of return of asset i; its square root $\sqrt{\sigma_{ii}}$ represents its *volatility*;
- $\sigma_{ij} > 0$ indicates that the rates of return of assets i and j tend to move in the same direction;
- $\sigma_{ij} < 0$ indicates that the rates of return of assets i and j tend to move in opposite directions;
- $\sigma_{ij} = 0$ indicates that the rates of return of assets i and j are uncorrelated.

Convention. *We identify every vector* $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ *with the column matrix*

$$\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix}.$$

The inner product between ξ and η we denote as

$$\langle \xi, \eta \rangle = \xi^T * \eta = \sum_{i=1}^{N} \xi_i \eta_i.$$

The following table presents examples of pairs of industries whose returns exhibit different signs of correlation.

Asset 1	**Asset 2**	**Typical relation**
Oil company	Petrochemical company	$\sigma_{12} > 0$
Steel producer	Automobile company	$\sigma_{12} > 0$
Bank	Insurance company	$\sigma_{12} > 0$
Oil company	Airline	$\sigma_{12} < 0$
Gold	Risky stocks	$\sigma_{12} < 0$ (sometimes)

The examples are illustrative. The sign of the empirical covariance between asset returns may change over time.

The portfolio is characterized by $(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ as in the previous subsection. The difference is that here the rate of returns of assets are random. We provide the following definitions.

Definition 7. *The **rate return of the portfolio** $x = (x_1, x_2, \ldots, x_N)$ is the random variable defined as follows:*

$$R_x = \sum_{i=1}^{N} x_i R_i.$$

*The **expected rate return of the portfolio** x is defined as follows:*

$$m_x = ER_x.$$

*The **variance of the portfolio** x is defined as follows:*

$$\sigma_x^2 = Var(R_x).$$

Remember the following facts:

Remember: For random variables $Y_1, Y_2, \ldots, Y_N$ such that $EY_i^2 < \infty$ for each i we have the following formula on variance:

$$Var\left(\sum_{i=1}^{N} Y_i\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} Cov(Y_i, Y_j) = \sum_{i=1}^{N} Var(Y_i) + 2\sum_{j<i} Cov(Y_i, Y_j).$$

Covariance is ***bilinear***, that is for given $\alpha \in \mathbb{R}$ and random variables Y and Z such that $EY^2 < \infty$ and $EZ^2 < \infty$:

$$Cov(\alpha Y, Z) = \alpha Cov(Y, Z) \quad \text{and} \quad Cov(Y, \alpha Z) = \alpha Cov(Y, Z).$$

Observe that the expected rate return of the portfolio $x = (x_1, x_2, \ldots, x_N)$ satisfies

$$m_x = ER_x = E\left(\sum_{i=1}^{N} x_i R_i\right) = \sum_{i=1}^{N} x_i ER_i = \sum_{i=1}^{N} x_i m_i = x^T * m. \tag{2}$$

In turn the variance is expressed as

$$\begin{aligned}
\sigma_x^2 &= Var(R_x) = Var\left(\sum_{i=1}^{N} x_i R_i\right) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} Cov(x_i R_i, x_j R_j) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j Cov(R_i, R_j) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} \\
&= x^T * \Sigma * x.
\end{aligned} \tag{3}$$

1.3.3 Optimal normalized portfolio

Consider the Markowitz model with random rates of return $(R_1, R_2, \ldots, R_N)$. The purpose of the investor is to construct a normalized portfolio $(x_1, x_2, \ldots, x_N)$, that is

$$\sum_{i=1}^{N} x_i = 1.$$

The ideal portfolio would be such that two conditions to be simultanously satisfied:

- The expected rate return R_x to be as large as possible;
- The variance or the rate return σ_x^2 to be as small as possible;

Because such portfolio does not exist, the investor is looking for a compromise solution according to the following **mean-variance criterion:**

$$\max \tau m_x - \sigma_x^2 \quad \text{such that} \quad \sum_{i=1}^{N} x_i = 1. \tag{4}$$

Here $\tau \geq 0$ is a *risk tolerance* parameter. Applying (2) and (3) the problem in (4) can be written as follows:

$$\max \tau \sum_{i=1}^{N} x_i m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}, \quad \text{such that} \quad \sum_{i=1}^{N} x_i = 1.$$

Definition 8. *For given $\tau \geq 0$, a normalized portfolio $(x_1^*, x_2^*, \ldots, x_N^*)$ is optimal in the* ***mean-variance*** *criterion if for every other normalized portfolio $(x_1, x_2, \ldots, x_N)$*

$$\tau \sum_{i=1}^{N} x_i^* m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i^* x_j^* \sigma_{ij} \geq \tau \sum_{i=1}^{N} x_i m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}.$$

The following examples make use of Kuhn's theorem, which we now state formally.

Assumption 1. *Suppose that X is an open convex subset of $\mathbb{R}^N$ and the following functions $g_1, g_2, \ldots, g_K$ and f are concave functions on X.*

Consider the following problem:

$$\max f(x) \quad \text{such that } g_1(x) \geq 0, g_2(x) \geq 0, \ldots, g_K(x) \geq 0, x \in X \tag{P}$$

and suppose that the following condition (called Slater's condition) holds:

$$\text{There is } y \in X \text{ such that } g_1(y) > 0, g_2(y) > 0, \ldots, g_K(y) > 0. \tag{S}$$

Let us denote $g(x) = (g_1(x), g_2(x), \ldots, g_K(x))$.

Theorem (Kuhn–Tucker)**.** *Suppose that Assumption 1 as well as the condition (S) holds. Then the following conditions are equivalent:*

(i) x^* *is a solution to problem (P);*

(ii) There exists a vector $\lambda \in \mathbb{R}^K$ *with nonnegative components (called Lagrange multipliers) such that* x^* *maximizes the Lagrangian*

$$\mathcal{L}(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle,$$

and the complementary slackness condition

$$\langle \lambda, g(x^*) \rangle = 0$$

holds.

If additionally $g_1, g_2, \ldots, g_K$ and f are differentiable then we can find x^* and λ from the

following system of equations:

$$\begin{cases} \frac{\partial f}{\partial x_1}(x) + \sum\limits_{k=1}^{K} \lambda_k \frac{\partial g_k}{\partial x_1}(x) &= 0 \\ \frac{\partial f}{\partial x_2}(x) + \sum\limits_{k=1}^{K} \lambda_k \frac{\partial g_k}{\partial x_2}(x) &= 0 \\ \vdots & \vdots \ \vdots \\ \frac{\partial f}{\partial x_N}(x) + \sum\limits_{k=1}^{K} \lambda_k \frac{\partial g_k}{\partial x_N}(x) &= 0 \\ \langle \lambda, g(x) \rangle &= 0 \end{cases}$$

Example 1. *Consider two stocks: the stock of an oil company and the stock of an airline. Their expected rates of return are* $m = (1, 3)$ *and their covariance matrix is*

$$\Sigma = \begin{pmatrix} 1 & -0.2 \\ -0.2 & 2 \end{pmatrix}.$$

Find the optimal normalized portfolios for $\tau = 0$ *and* $\tau = 1$.

Solution. *We solve this problem elementary. The mean-variance criterion is*

$$\max \tau(x_1 + 3x_2) - x_1^2 + 0.2x_1x_2 + 0.2x_2x_1 - 2x_2^2 \quad \textit{such that} \quad x_1 + x_2 = 1.$$

Since $x_2 = 1 - x_1$, *then if we denote* $x_1 \equiv x$ *the problem leads to*

$$\max \tau(3 - 2x) - x^2 - 2(1 - x)^2 + 0.4x(1 - x) \quad \textit{where} \quad x \in \mathbb{R}.$$

Let $\tau = 0$. *Then we solve*

$$\max(-x^2 - 2(1 - x)^2 + 0.4x(1 - x)) = \max(-3.4x^2 + 4.4x - 2).$$

We have the standard quadratic function. Hence the optimal value is reached at

$$x^* = \frac{-4.4}{2(-3.4)} = \frac{4 + \frac{4}{10}}{2(3 + \frac{4}{10})} = \frac{44}{2 * 34} = \frac{22}{34} = \frac{11}{17}.$$

Then $x_1^* = x^* = \frac{11}{17}$ *and* $x_2^* = 1 - x_1^* = \frac{6}{17}$. *Hence if the investor has* 0 *risk tolerance the*

optimal portfolio if $\left(\frac{11}{17}, \frac{6}{17}\right) = (64.71\%, 63.29\%)$. *As a result, such investor invest* 64.71% *on the oil company and the rest on the stock of an airplain.*

Let $\tau = 1$. *Then we solve*

$$\max((x + 3(1-x)) - x^2 - 2(1-x)^2 + 0.4x(1-x)) = \max(-3.4x^2 + 2.4x + 1).$$

We have again the standard quadratic function. Hence the optimal value is reached at

$$x^* = \frac{-2.4}{2(-3.4)} = \frac{2.4}{2(3.4)} = \frac{2 + \frac{4}{10}}{2\left(3 + \frac{4}{10}\right)} = \frac{12}{34}.$$

Then $x_1^* = x^* = \frac{12}{34}$ *and* $x_2^* = 1 - x_1^* = \frac{22}{34}$. *Hence if the investor has* 1 *risk tolerance the optimal portfolio if* $\left(\frac{12}{34}, \frac{22}{34}\right) = (35.29\%, 64.71\%)$. *As a result, such investor invest* 35.29% *on the oil company and the rest on the stock of an airplain.*

Example 2. *Consider two stocks: the stock of a steel producer and the stock of an automobile manufacturer. Their expected rates of return are* $m = (1, 3)$ *and their covariance matrix is*

$$\Sigma = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 2 \end{pmatrix}.$$

Find the optimal normalized portfolios for $\tau = 0$ *and* $\tau = 1$.

Solution. *We solve this problem elementary. The mean-variance criterion is*

$$\max \tau(x_1 + 3x_2) - x_1^2 - 0.2x_1x_2 - 0.2x_2x_1 - 2x_2^2 \quad \textit{such that} \quad x_1 + x_2 = 1.$$

Since $x_2 = 1 - x_1$, *then if we denote* $x_1 \equiv x$ *the problem leads to*

$$\max \tau(3 - 2x) - x^2 - 2(1-x)^2 - 0.4x(1-x) \quad \textit{where} \quad x \in \mathbb{R}.$$

Let $\tau = 0$. *Then we solve*

$$\max(-x^2 - 2(1-x)^2 - 0.4x(1-x)) = \max(-2.6x^2 + 3.6x - 2).$$

Once again we have the standard quadratic function. Hence the optimal value is reached at

$$x^* = \frac{3.6}{2 * (2.6)} = \frac{3 + \frac{6}{10}}{2\left(2 + \frac{6}{10}\right)} = \frac{36}{2 * 26} = \frac{18}{26} = \frac{9}{13}.$$

Then $x_1^* = x^* = \frac{9}{13}$ *and* $x_2^* = 1 - x_1^* = \frac{4}{13}$. *Hence if the investor has* 0 *risk tolerance the optimal portfolio if* $\left(\frac{9}{13}, \frac{4}{13}\right) = (69.23\%, 30.77\%)$. *As a result, such investor invest* 69.23% *on the stock of a steel producer and the rest on the stock of an automobile manufacture.*

Let $\tau = 1$. *Then we solve*

$$\max((x + 3(1 - x)) - x^2 - 2(1 - x)^2 - 0.4x(1 - x)) = \max(-2.6x^2 + 1.6x + 1).$$

We have once again the standard quadratic function. Hence the optimal value is reached at

$$x^* = \frac{1.6}{2(2.6)} = \frac{1 + \frac{6}{10}}{2\left(2 + \frac{6}{10}\right)} = \frac{8}{26} = \frac{4}{13}.$$

Then $x_1^* = x^* = \frac{4}{13}$ *and* $x_2^* = 1 - x_1^* = \frac{9}{13}$. *Hence if the investor has* 1 *risk tolerance the optimal portfolio if* $\left(\frac{4}{13}, \frac{9}{13}\right) = (30.77\%, 69.23\%)$. *As a result, such investor invest* 30.77% *on the stock of a steel producer and the rest on the stock of an automobile manufacture.*

1.3.4 Optimal self-financing portfolio

Consider the Markowitz model with random rates of return $(R_1, R_2, \ldots, R_N)$. The purpose of the investor is to construct a self-financing portfolio $(x_1, x_2, \ldots, x_N)$, that is

$$\sum_{i=1}^{N} x_i = 0.$$

Similarly as in the case of normalized portfolio, the investor is looking for a compromise solution according to the following **mean-variance criterion:**

$$\max \tau m_x - \sigma_x^2 \quad \text{such that} \quad \sum_{i=1}^{N} x_i = 0. \tag{5}$$

Here $\tau \geq 0$ is a *risk tolerance* parameter. Applying (2) and (3) the problem in (5) can be written as follows:

$$\max \tau \sum_{i=1}^{N} x_i m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}, \quad \text{such that} \quad \sum_{i=1}^{N} x_i = 0.$$

Definition 9. *For given $\tau \geq 0$, a self-financing portfolio $(x_1^*, x_2^*, \ldots, x_N^*)$ is optimal in the **mean-variance** criterion if for every other self-financing portfolio $(x_1, x_2, \ldots, x_N)$*

$$\tau \sum_{i=1}^{N} x_i^* m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i^* x_j^* \sigma_{ij} \geq \tau \sum_{i=1}^{N} x_i m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}.$$

Example 3. *Consider two stocks: the stock of a steel producer and the stock of an automobile manufacturer. Their expected rates of return are $m = (1, 3)$ and their covariance matrix is*

$$\Sigma = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 2 \end{pmatrix}.$$

Find the optimal self-financing portfolios ragarding to $\tau \geq 0$.

Solution. *We solve this problem elementary. The mean-variance criterion is*

$$\max \tau(x_1 + 3x_2) - x_1^2 - 0.2x_1x_2 - 0.2x_2x_1 - 2x_2^2 \quad \textit{such that} \quad x_1 + x_2 = 0.$$

Since $x_2 = -x_1$, then if we denote $x_1 \equiv x$ the problem leads to

$$\max(-2\tau x - 2.6x^2) \quad \textit{where} \quad x \in \mathbb{R}.$$

Once again we have the standard quadratic function. Hence the optimal value is reached at

$$x^* = -\frac{2}{2 * (2.6)}\tau = -\frac{1}{2 + \frac{6}{10}}\tau = -\frac{10}{26}\tau = -\frac{5}{13}\tau = -\frac{5}{13}\tau.$$

Then $x_1^* = x^* = -\frac{5}{13}\tau$ and $x_2^* = -x_1^* = \frac{5}{13}\tau$. As a result, the investor does not invest in such a portfolio if his risk tolerance is $\tau = 0$. Otherwise, he sells the stock of the

steel company (short position in the stock of the steel company) to buy the stock of the automobile company (long position in the stock of the automobile company). The higher his risk tolerance, the more stock of the steel company he sells and, consequently, the more stock of the automobile company he buys.

1.3.5 Solution of the problem - Lagrange multipliers

When there are more than one 2 assets, the elementary solution of maximization of mean-variance problem is problematic.

Example 4. *Consider three assets: the stock of an oil company, the stock of a petrochemical company, and the stock of an airline. Their covariance matrix is given by*

$$\Sigma = \begin{bmatrix} 1 & 0.2 & -0.2 \\ 0.2 & 2 & -0.1 \\ -0.2 & -0.1 & 1.5 \end{bmatrix}.$$

Interpret the covariance matrix Σ. In particular:

1. *compare the risk associated with the three assets;*
2. *identify which pairs of assets tend to move in the same direction;*
3. *identify which pairs of assets tend to move in opposite directions;*
4. *discuss which assets may provide diversification benefits when included in the same portfolio;*
5. *Find optimal normalized portfolio whenever $\tau = 0$.*

Solution. *Adnotations below:*

A1 The risks are reflected on the diagonal of Σ where is variances. Hence the variance of the stock of an oil company is $\sigma_{11} = 1$, the variance of the stock of a petrochemical company is $\sigma_{22} = 2$ and the variance of the stock of an airline is $\sigma_{33} = 1.5$. The most risky is the stock of a petrochemical company, and the less risky is the stock of an oil company;

A2 *Because* $\sigma_{12} = \sigma_{21} = 0.2 > 0$, *the stock of an oil company and stock of a petrochemical company tend to move in the same direction;*

A3 *Because* $\sigma_{13} = \sigma_{13} = -0.2 < 0$, *the stock of an oil company and the stock of the airline tend to move in opposite directions. Moreover, because* $\sigma_{23} = \sigma_{23} = -0.1 < 0$, *the stock of a petrochemical company and the stock of the airline tend to move in opposite directions;*

A4 *By A3, the stock of an oil company and the stock of an airline may provide diversification benefits, since a potential loss on the stock of the oil company may be partially offset by a gain on the stock of the airline. By the same argument, combining the stock of a petrochemical company with the stock of an airline may also provide diversification benefits.*

A5 *We have to maximize*

$$\begin{aligned} -x^T \Sigma x &= -(x_1^2 + 2x_2^2 + 1.5x_3^2 + 0.4x_1x_2 - 0.4x_1x_3 - 0.2x_2x_3) \\ &= -x_1^2 - 2x_2^2 - 1.5x_3^2 - 0.4x_1x_2 + 0.4x_1x_3 + 0.2x_2x_3. \end{aligned}$$

such that $x_1 + x_2 + x_3 = 1$. *The Lagrange function is*

$$\mathcal{L}(x, \lambda) = -x_1^2 - 2x_2^2 - 1.5x_3^2 - 0.4x_1x_2 + 0.4x_1x_3 + 0.2x_2x_3 + \lambda(1 - x_1 - x_2 - x_3).$$

We solve

$$\left\{ \begin{array}{rcrcl} \frac{\partial \mathcal{L}}{\partial x_1}(x, \lambda) &=& -2x_1 - 0.4x_2 + 0.4x_3 - \lambda &=& 0 \\ \frac{\partial \mathcal{L}}{\partial x_2}(x, \lambda) &=& -4x_2 - 0.4x_1 + 0.2x_3 - \lambda &=& 0 \\ \frac{\partial \mathcal{L}}{\partial x_3}(x, \lambda) &=& -3x_3 + 0.4x_1 + 0.2x_2 - \lambda &=& 0 \\ & & x_1 + x_2 + x_3 &=& 1. \end{array} \right.$$

The optimal solution in case of 0 *risk tolerance is*

$$(x_1^*, x_2^*, x_3^*) = \left(\frac{309}{673}, \frac{124}{673}, \frac{240}{673}\right) = (45.91\%, 18.42\%, 35.66\%).$$

In this example, we see that the riskier the asset, the smaller its proportion in the optimal portfolio.

1.4 Eficient portfolios

Let $e := [1, 1, \ldots, 1]^T \in \mathbb{R}^N$. The Markowitz optimization problem can be written in a concise way:

$$M_\tau : \max \tau m_x - \sigma_x^2 \quad \text{such that} \quad \langle e, x \rangle = 1$$

in case of normalized portfolio. In case of self financing portfolio the subject is $\langle e, x \rangle = 0$. We focus attention for normalized portfolios. The solution of Markowitz problem belongs to larger class of portfolios.

Definition 10. *We say that a normalized portfolio $x^* \in \mathbb{R}^N$ is **efficient** if x^* solves M_τ for some $\tau \geq 0$. The plot (m_x, σ_x^2) where x runs through efficient portfolios is called an **efficiency frontier**.*

We introduce two assumptions.

Assumption 2. *The covariance matrix Σ is positive definite.*

Assumption 3. *There is at least two $i \neq j$ such that $m_i \neq m_j$.*

We demonstrate later that Assumption 2 guarantees that there are no risk-free portfolios with strictly positive returns.

We also demonstrate that if Assumption 3 fails, the efficient frontier degenerates; that is, there is only one efficient portfolio, namely the minimum-variance portfolio.

The following theorem is useful. The proof is elementary and we leave it.

Theorem 1. *If $\Sigma \in \mathbb{R}^N$ is positive definite then $f(x) = -x^T * \Sigma * x$ is a strictly concave function on $\mathbb{R}^N$.*

1.4.1 Discussion of the assumptions

Recall that $\sigma_{ij} = Cov(R_i, R_j)$. The following remarks on Assumptions (2) and (3) are worth mentioning.

If Assumption 2 fails, then there is $(y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$ such that

$$y_1 R_1(\omega) + y_2 R_2(\omega) + \ldots + y_N R_N(\omega) = c \quad \text{for some } c \in \mathbb{R}, \text{ for } \omega - a.a.\omega \in \Omega, \tag{6}$$

which is equivaletnt to

$$Var\left(\sum_{i=1}^{N} y_i R_i\right) = y^T * \Sigma * y = 0. \tag{7}$$

Then we have the following alternative consequences:

- If $c = 0$ then there is a non zero portfolio $y = (y_1, y_2, \ldots, y_N)$, not necessarily normalized, such that (6) leads to the linear dependences of $R_1, R_2, \ldots, R_N$, that is

  $$\langle R, y\rangle = y_1 R_1(\omega) + y_2 R_2(\omega) + \ldots + y_N R_N(\omega) = 0 \quad \text{for } P - a.a.\omega \in \Omega.$$

  Furthermore, for some i, $y_i \neq 0$. Then P-almost surely we have

  $$R_i(\omega) = -\sum_{j \neq i} \frac{y_j}{y_i} R_j(\omega)$$

  and the asset i is redundant in this portfolio;

- If $c > 0$ then the portfel y is risk free and from (6) has strictly positive return c;

- If $c < 0$ then we can substitute $\tilde{y} := -y$ and this portfolio $\tilde{y}$ is still risk-free and has strictly positive return $-c$;

If Assumption 3 fails, then all assets have the same expectation and $m = (\theta, \theta, \ldots, \theta) = \theta \cdot e$ for some θ. Thus,

$$\tau \langle m, x\rangle = \langle \theta \cdot e, x\rangle = \theta \langle e, x\rangle = \theta \sum_{i=1}^{N} x_i = \theta$$

since the portfolio x is normalized. Thus, in every M_τ-problem leads to

$$M_\tau : \max \tau\theta - x^T \Sigma x \quad \text{such that } \langle e, x\rangle = 1.$$

Indeed, in M_τ all objective functions differ only by a constant, and hence they attain their optimal values at the same point.

Remark 2. *In particular, if* $\sum_{i=1}^{N} y_i \neq 0$*, then using substitutions*

$$\tilde{y}_i := \frac{y_i}{\sum_{i=1}^{N} y_i} \quad \textit{and} \quad \tilde{c} := \frac{c}{\sum_{i=1}^{N} y_i},$$

then coexists a normalized portfolio $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_N)$ *such that (6) and (7) holds c.*

1.4.2 Main characterization of efficient portfolio

The following result is a characterization of an efficient portfolio.

Theorem 2. *Suppose that Assumption 2 holds. A normalized portfolio* x^* *is efficient if and only if is Pareto optimal, that is, there is no normalized portfolio such that*

$$m_x \geq m_{x^*} \quad \textit{and} \quad \sigma_x^2 \leq \sigma_{x^*}^2 \tag{8}$$

with at least one strict inequality.

Proof. Suppose that x^* is efficient. On the contrary, suppose that there is a normalized portfolio for which $m_x \geq m_{x^*}$ and $\sigma_x^2 \leq \sigma_{x^*}^2$ with at least one strict inequality. Hence for every $\tau > 0$

$$\tau m_x - \sigma_x^2 > \tau m_{x^*} - \sigma_{x^*}^2.$$

Thus, x^* to preserve the efficiency, would be 0-efficient. Then, by assumption on x, we would have $\sigma_{x^*}^2 = \sigma_x^2$ and $m_x > m_{x^*}$. Define a portfolio $z = \frac{1}{2}x^* + \frac{1}{2}x$. Then

$m_z = \frac{1}{2}m^* + \frac{1}{2}m_x > m_{x^*}$, and

$$\sigma_z^2 = Var\left(\sum_{i=1}^{N}\frac{1}{2}(x_i^* + x_i)R_i\right) = \frac{1}{2}(x+x^*)^T\Sigma\frac{1}{2}(x+x^*) := g\left(\frac{1}{2}(x+x^*)\right).$$

Applying Theorem 1 to $\varphi(\cdot) = \sigma_x^2$ we know that $\varphi(\cdot)$ is strictly convex, hence

$$\sigma_z^2 = \varphi\left(\frac{1}{2}(x+x^*)\right) < \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x^*) = \frac{1}{2}(\sigma_x^2 + \sigma_{x^*}^2) = \sigma_{x^*}^2.$$

As a result $\sigma_z^2 < \sigma_{x^*}^2$ and $m_z > m_{x^*}$. Hence x^* does not even qualify as 0-efficient. This contradiction means that (8) holds.

Suppose that (8) holds. Than means that x^* maximizes the function

$$\varphi(x) = -\sigma_x^2 \quad \text{such that } m_x \geq m^*, \text{ and } \langle e, x\rangle = 1,$$

where $m^* := m_{x^*}$. By Assumption 2 and Theorem 1 applied to $\varphi(x)$ we easily conclude that $\varphi(\cdot)$ is a concave twice continuously differentiable function. Applying Kuhn-Tucker Theorem, there are Lagrange multipliers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that x^* maximizes the following function

$$\mathcal{L}(x,\lambda) = -\sigma_x^2 + \lambda_1(m_x - m^*) + \lambda_2(1 - \langle e, x\rangle).$$

Thus, for each normalized portfolio $x \in \mathbb{R}^N$,

$$-\sigma_x^2 + \lambda_1(m_x - m^*) \leq -\sigma_{x^*}^2 + \lambda_1(m_{x^*} - m^*),$$

hence

$$\lambda_1 m_x - \sigma_x^2 \leq \lambda_1 m_{x^*} - \sigma_{x^*}^2.$$

Taking $\tau := \lambda_1$, we obtain that x^* is M_τ efficient. □

References

[1] Evstigneev, I.V, Hens, T., Schenk-Hoppé, K.R. (2015). *Mathematical Financial Economics. A Basic Introduction*. Springer Cham Heidelberg New York Dordrecht London © Springer International Publishing Switzerland.