Problem Set 1. The Markowitz Portfolio Model

1. Consider two risky assets with covariance matrix

$$\Sigma = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}.$$

   (a) What are the variances of the returns of the two assets?

   (b) What are their standard deviations?

   (c) What is the covariance between the returns?

   (d) Calculate the correlation coefficient.

   (e) Which asset is more risky when considered separately?

   (f) What does the negative covariance tell us about the movements of the two returns?

   (g) Explain why combining these assets may reduce portfolio risk.

2. Consider the following covariance matrices:

$$\Sigma_1 = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}, \qquad \Sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Sigma_3 = \begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix}.$$

   For each matrix consider the normalized portfolio

$$x = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

   (a) Calculate the variance of the portfolio in each case.

   (b) Compare the three results.

   (c) Explain the role of covariance in portfolio diversification.

   (d) In which case are the diversification benefits the largest?

3. Consider two risky assets with expected returns

$$m = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and covariance matrix

$$\Sigma = \begin{bmatrix} 1 & -0.2 \\ -0.2 & 2 \end{bmatrix}.$$

Consider a normalized portfolio

$$x_1 + x_2 = 1, \qquad x_1, x_2 \geq 0.$$

(a) Write down the expected return and variance of the portfolio.

(b) Find the minimum-variance portfolio.

(c) Solve the Markowitz problem

$$\max_{x_1, x_2} \left\{ \tau m_x - \sigma_x^2 \right\}$$

for $\tau = 2$.

(d) Interpret the optimal portfolio.

4. An investor considers stocks of an oil company and an airline. Their expected returns are

$$m = \begin{bmatrix} 0.08 \\ 0.12 \end{bmatrix},$$

and their covariance matrix is

$$\Sigma = \begin{bmatrix} 0.04 & -0.02 \\ -0.02 & 0.09 \end{bmatrix}.$$

The investor allocates all wealth between the two stocks and short selling is not allowed.

(a) Explain economically why the returns of an oil company and an airline may be negatively correlated.

(b) Which stock is more risky when considered separately?

(c) Write down the expected return and variance of a normalized portfolio (x_1, x_2).

(d) Find the minimum-variance portfolio subject to

$$x_1 + x_2 = 1, \qquad x_1, x_2 \geq 0.$$

(e) Calculate the expected return of this portfolio.

(f) Explain the economic source of the diversification effect.

5. An investor considers stocks of two automobile producers operating in the same market. Their expected returns are

$$m = \begin{bmatrix} 0.08 \\ 0.12 \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.09 \end{bmatrix}.$$

(a) Explain why the covariance between the two returns may be positive.

(b) Find the minimum-variance normalized portfolio

$$x_1 + x_2 = 1, \qquad x_1, x_2 \geq 0.$$

(c) How would the portfolio variance change if the covariance were equal to -0.03 instead of 0.03?

(d) Explain the difference in economic terms.

6. Consider two assets with

$$m = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & -0.2 \\ -0.2 & 2 \end{bmatrix}.$$

Short selling is allowed. Thus,

$$x_1 + x_2 = 1, \qquad x_1, x_2 \in \mathbb{R}.$$

Consider

$$\max_x \left\{ \tau m_x - \sigma_x^2 \right\}.$$

(a) Reduce the problem to an optimization problem in one variable.

(b) Find the optimal portfolio as a function of $\tau > 0$.

(c) Determine for which values of τ the optimal portfolio contains a short position.

(d) Explain why a sufficiently risk-tolerant investor may choose a portfolio with a negative weight.

7. Consider two assets with

$$m = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 2 \end{bmatrix}.$$

The investor constructs a self-financing portfolio:

$$x_1 + x_2 = 0.$$

Consider the problem

$$\max_{x \in \mathbb{R}^2} \left\{ \tau m_x - \sigma_x^2 \right\}, \qquad \tau > 0.$$

(a) Use $x_2 = -x_1$ to reduce the problem to one variable.

(b) Find the optimal portfolio as a function of τ.

(c) Which asset is held long and which asset is sold short?

(d) Explain the economic meaning of the condition

$$x_1 + x_2 = 0.$$

8. Two oil companies, A and B, operate in the same market and are exposed to similar changes in the world price of oil. Their expected returns are

$$m = \begin{bmatrix} 0.06 \\ 0.09 \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} 0.04 & 0.035 \\ 0.035 & 0.04 \end{bmatrix}.$$

An investor believes that company B will outperform company A and constructs the self-financing portfolio

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(a) Explain how this portfolio can be constructed without an initial net investment.

(b) Calculate its expected return.

(c) Calculate its variance.

(d) Why does the high positive covariance between the two stocks reduce the risk of this long–short position?

(e) What economic risk has been largely eliminated by combining two companies from the same industry?

(f) What risk is the investor primarily taking?

9. Consider three risky assets with expected returns

$$m = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0.2 & -0.2 \\ 0.2 & 2 & 0.1 \\ -0.2 & 0.1 & 1.5 \end{bmatrix}.$$

Consider normalized portfolios satisfying

$$x_1 + x_2 + x_3 = 1, \qquad x_i \geq 0.$$

(a) Interpret all diagonal and off-diagonal elements of Σ.

(b) Which asset has the highest individual risk?

(c) Which pair of assets appears particularly attractive from the point of view of diversification?

(d) Write down the expected return and variance of the portfolio.

(e) Formulate the Markowitz optimization problem.

(f) Write down the Kuhn–Tucker conditions.

(g) Find the minimum-variance portfolio.

(h) Determine whether all three assets are included in the optimal portfolio.

10. An investor considers stocks of three companies: an electricity producer, an automobile producer, and an oil company. Their expected returns are

$$m = \begin{bmatrix} 0.04 \\ 0.09 \\ 0.07 \end{bmatrix},$$

and their covariance matrix is

$$\Sigma = \begin{bmatrix} 0.02 & 0.01 & 0 \\ 0.01 & 0.08 & -0.02 \\ 0 & -0.02 & 0.05 \end{bmatrix}.$$

(a) Interpret every diagonal element of Σ.

(b) Interpret the signs of all off-diagonal elements.

(c) Which stock is the most risky when considered separately?

(d) Which pair of stocks provides the strongest diversification effect according to the covariance matrix?

(e) Give an economic explanation for the negative covariance between the automobile producer and the oil company.

(f) Find the minimum-variance normalized portfolio subject to

$$x_1 + x_2 + x_3 = 1, \qquad x_i \geq 0.$$

(g) Compare the risk of the optimal portfolio with the risks of the three individual stocks.

11. Consider three assets with

$$m = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & 0.7 & 0.6 \\ 0.7 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}.$$

A self-financing portfolio satisfies

$$x_1 + x_2 + x_3 = 0.$$

(a) Give an example of a nonzero self-financing portfolio.

(b) Calculate its expected return and variance.

(c) Solve

$$\max_x \left\{ \tau m_x - \sigma_x^2 \right\}$$

subject to

$$x_1 + x_2 + x_3 = 0.$$

(d) Interpret the signs of the optimal portfolio weights.

(e) Explain why a self-financing portfolio necessarily involves both long and short positions unless it is the zero portfolio.